A NOTE ON RINGS WITH A WEAK ALGORITHM

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Introduction. The principal examples of rings with a weak algorithm in [3] are tensor rings on K-bimodules, K a division ring. By introducing two maps S: $K \to K_n$, D: $K \to V_n(K)$ which satisfy the Ore-like conditions: $(\alpha\beta)^S = \alpha^S\beta^S$, $(\alpha\beta)^D = \alpha^D\beta^S + \alpha\beta^D$, all α, β in K, various rings with the weak algorithm are obtained from tensor rings. Here, $n = |X|$, X a right K-basis of the given K-bimodule N. A separation of this class into families is easily observed. To deal with arbitrary rings with a weak algorithm, in §2 a class of rings extending the class of tensor rings is defined. As for tensor rings, other rings with the weak algorithm (all of them, in fact) are obtained from these rings and separated into families. It is shown then that each ring in the class extending the class of tensor rings is embeddable in a division ring and this raises a central question: Can the embedding theorem be generalized so as to apply to the associated classes?

Throughout, ring shall mean associative ring with 1, K shall denote a division ring, X a generating set for the ring and x_I the monomial $x_{i_1} \cdots x_{i_n}$ where $I = (i_1, \cdots, i_n)$. Although the proofs are carried out using right polynomials, right K-modules, etc., the results in [3] insure that there is a left-right symmetry.

1. Families associated with tensor rings. Let R be a ring with the weak algorithm, let K be the underlying division ring and let $X = \{x_\lambda : \lambda \in \Lambda\}$ be an R-independent generating set of R. We set $N = \{\sum x_\lambda \alpha_\lambda : x_\lambda \in X\}$ and for the moment restrict ourselves to rings for which $M = N + K$ is a K-bimodule. Our first result is

THEOREM 1.1 *Let R be a ring with a weak algorithm for which M is a K-bimodule. The structure of M, and hence of R, is completely determined by two maps S: $K \to K_n$, D: $K \to V_n(K)$, $n = |Z|$, Z a right K-basis of N, which satisfy $(\alpha\beta)^S = \alpha^S\beta^S$, $(\alpha\beta)^D = \alpha^D\beta^S + \alpha\beta^D$, $K \approx K^S$. Moreover, any such S, D yield a ring with the weak algorithm.*

As each element of R has a unique expression

(1) $\sum x_I\alpha_I$ (a.a. $\alpha_I = 0$, I over all ordered finite subsets of Λ)

as a right polynomial on X, the structure of M determines that of R. Next order X: $\{x_1, x_2, \cdots\}$ and for each x_i set