# CHARACTERISTIC CLASSES–OLD AND NEW[1,2]

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**1. Definition of sphere bundles.** Let $M^n$ be an $n$-dimensional, $C^\infty$-manifold. Define $T(M)$ to be all vectors tangent to $M$ of unit length. Define $p: T(M)\rightarrow M$ by $p$(vector) = initial point of the vector. Then $p$ is a continuous function with $p^{-1}(m)$ homeomorphic to $S^{n-1}$ if $m\in M$. $(T(M), p, M)$ is an example of an $(n-1)$-sphere bundle.

Let me now abstract some of the properties of this example and define an $(n-1)$-sphere bundle. An $(n-1)$-sphere bundle $\xi$ is a triple $(E, p, X)$, where $p: E\rightarrow X$ is a continuous function, $X$ has a covering by neighborhoods $\{V_\alpha\}$ such that $h_\alpha: p^{-1}(V_\alpha)\rightarrow V_\alpha\times S^{n-1}$, where $h$ is a homeomorphism, $h_\alpha(e)=(p(e), S_\alpha(e))$. That is, we can give coordinates to $p^{-1}(V_\alpha)$ using $V_\alpha$ and $S^{n-1}$. Furthermore, there is a condition on changing coordinates; namely, if $e\in p^{-1}(V_\alpha\cap V_\beta)$, then $h_\alpha(e)=(p(e), S_\alpha(e))$ and $h_\beta(e)=(p(e), S_\beta(e))$ and we obtain a function $S_\beta^\alpha: S^{n-1}\rightarrow S^{n-1}$ given by $S_\beta^\alpha(S_\alpha(e))=S_\beta(e)$, defined for each $p(e)\in V_\alpha\cap V_\beta$. We demand that $S_\beta^\alpha\in O(n)$, the orthogonal group of homeomorphisms of $S^{n-1}$. Finally, $S_\beta^\alpha$ depends on $p(e)$ and this dependence must be continuous.

Two $(n-1)$-sphere bundles $\xi$ and $\eta$ over $X$ are called equivalent if there is a homeomorphism $F: E_\xi\rightarrow E_\eta$ such that

$$\begin{array}{ccc} E_\xi & \xrightarrow{F} & E_\eta \\ & p\searrow \quad \swarrow p & \\ & X & \end{array}$$

commutes and such that $F|p^{-1}(x)\in O(n)$ for all coordinates on $p^{-1}(x)$.

A very important example of an $(n-1)$-sphere bundle is the following one. Let $\mathrm{BO}(n)$ = the Grassmann space of all $n$-planes through the origin in $R^\infty$. Let $\mathrm{EO}(n)$ be the set of pairs, an element of $\mathrm{BO}(n)$ and a unit vector in that $n$-plane. Let $p: \mathrm{EO}(n)\rightarrow\mathrm{BO}(n)$ be the first element of the pair. The importance of this example is shown by the following classification theorem.

[1] An address delivered before the New York meeting of the Society by invitation of the Committee to Select Hour Speakers, April 13, 1968; received by the editors April 21, 1969.

[2] In order not to obscure the structure of the subject, I have left out a number of technicalities; in fact some of the statements may be incorrect as stated.