SOME NEW BOUNDS ON PERTURBATION OF SUBSPACES

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When a Hermitian linear operator A is slightly perturbed, by how much can its invariant subspaces change? Given some approximations to a cluster of neighboring eigenvalues and to the corresponding eigenvectors of a real symmetric matrix, and given a lower bound $\delta > 0$ for the gap that separates the cluster from all other eigenvalues, how much can the subspace spanned by the eigenvectors differ from that spanned by our approximations? These questions are closely related; both are investigated here. First the difference between the two subspaces is characterized in terms of certain angles through which one subspace must be rotated in order most directly to reach the other. The angles constitute the spectrum of a Hermitian operator Θ, with which is associated a commuting skew-Hermitian operator $J = -J^3$; the unitary operator that differs least from the identity and rotates one subspace into the other turns out to be $\exp(J\Theta)$. These operators unify the treatment of natural geometric, operator-theoretic and error-analytic questions concerning those subspaces. Given the gap δ, and given bounds upon either the perturbation (1st question) or a computable residual (2nd question), we obtain sharp bounds upon unitary-invariant norms of trigonometric functions of Θ. (A norm is unitary-invariant whenever $\|L\| = \|ULV\|$ for all unitary U and V. Examples are the bound-norm $\|L\|_1 \equiv \sup \|Lx\| / \|x\|$ and the square-norm $\|L\|_{sq} \equiv (\operatorname{trace} L^*L)^{1/2}$.)

In this note we consider a finite-dimensional unitary space $\mathcal{H}$ in which the scalar product is denoted by y^*x, and $\|x\| \equiv (x^*x)^{1/2}$. Proofs of the following statements will appear elsewhere, together with extensions to infinite-dimensional Hilbert spaces and to noncompact or unbounded operators [2]. That article discusses the relation of our results to earlier work on the subject, such as [1], [3], [4].

1. **Subspaces and isometries.** There are two convenient ways to identify a subspace of $\mathcal{H}$. On the one hand, let P be the orthogonal projector ($P = P^* = P^2$) onto that subspace, which is then denoted by $P\mathcal{H}$. On the other hand, let $e_1, e_2, \cdots, e_n$ be an orthonormal basis for the subspace; then $E \equiv (e_1, e_2, \cdots, e_n)$ denotes an isometry mapping column n-vectors into the subspace, which is now interpreted as the range $\mathcal{R}(E)$. Orthonormality of the e_i's means $E^*E = ((e_i^*e_j)) = 1$, the identity operator on the n-space; $\mathcal{R}(E) = P\mathcal{H}$ means $P = EE^* = \sum_1^n e_ie_i^*$. Note that the subspace does not determine E