ON INJECTIVE BANACH SPACES AND THE SPACES $C(S)$

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A Banach space is injective (resp. a $\mathcal{P}_1$ space) if every isomorphic (resp. isometric) imbedding of it in an arbitrary Banach space Y is the range of a bounded (resp. norm-one) linear projection defined on Y.

In §1 we study linear topological properties of injective Banach spaces and the spaces $C(S)$ themselves; in §2 we study their conjugate spaces. (Throughout, "S" denotes an arbitrary compact Hausdorff space.) For example, by applying a result of Gaifman [**3**], we obtain in §1 that there exists a $\mathcal{P}_1$ space which is not isomorphic to any conjugate Banach space. We also obtain there that S satisfies the countable chain condition (the C.C.C.) if and only if every weakly compact subset of $C(S)$ is separable. (S is said to satisfy the C.C.C. if every uncountable family of open subsets of S contains two distinct sets with nonempty intersection.) In §2 we classify up to isomorphism (linear homeomorphism) all the conjugate spaces (B^*, B^{**}, B^{***}, etc.) of the $\mathcal{P}_1$ spaces $B=L^\infty(\mu)$ for some finite measure μ, or $B=l^\infty(\Gamma)$ for some infinite set Γ. (The isomorphic classification of the spaces $L^\infty(\mu)$ for finite measures μ is given in [**8**].) We also determine in §2 the injective quotients of the above spaces B, and show that every injective Banach space of dimension the continuum, has its dual isomorphic to $(l^\infty)^*$. (Dimension of a Banach space Y (denoted dim Y) equals the minimum of the cardinalities of subsets of Y with dense linear span.)

We include some of the proofs; full details of these and other results will appear in [**7**].

1. We say that S carries a *strictly positive measure* if there exists a $\mu \in M(S)$ (the space of bounded Radon measures on S) such that $\mu(U)>0$ for all nonempty open $U\subset S$.

THEOREM 1.1. *Let S satisfy the C.C.C. and suppose that $C(S)$ is isomorphic to a conjugate Banach space. Then S carries a strictly positive measure.*

PROOF. The hypotheses and the Riesz representation theorem imply that there exists a closed subspace A of $M(S)$ such that $C(S)$ is isomorphic to A^* and A is weak* dense in $M(S)$ (identifying $M(S)$

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