APPLICATIONS OF AFFINE ROOT SYSTEMS TO THE THEORY OF SYMMETRIC SPACES[1]

BY LAWRENCE CONLON

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Introduction. Let $(G;\ K_1,\ K_2)$ be a compact symmetric triad in the sense of [3], G simply connected. The natural action of K_1 on G/K_2 is of interest because it is variationally complete [5]. In [3] we introduced certain "affine root systems" in order to describe the orbits of this K_1-action, and in the piesent note we wish to announce the classification [4] of these systems and to indicate further applications to the theory of symmetric spaces.

1. **Preliminaries.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra, ν an automorphism of $\mathfrak{g}$, and set $\mathfrak{g}_\nu = \{X \in \mathfrak{g} : \nu(X) = X\}$. The following is due essentially to de Siebenthal [7] (cf. also [4, §7]).

(1.1) Proposition. *If $\mathfrak{h}_\nu \subset \mathfrak{g}_\nu$ is a Cartan subalgebra, there is a unique Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{h}_\nu \subset \mathfrak{h}$. There is a finite family $\mathfrak{a} = \{\zeta : \mathfrak{h}_\nu \to \boldsymbol{C}/i\boldsymbol{Z}\}$ of affine functionals and an orthogonal direct sum decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \sum \mathfrak{g}_\zeta, \qquad \zeta \in \mathfrak{a}$$

where $\dim(\mathfrak{g}_\zeta) = 1$ *and*

$$\nu \circ \exp(\mathrm{ad}(Z)) \mid \mathfrak{g}_\zeta = \exp(2\pi\zeta(Z)),$$

for all $Z \in \mathfrak{h}_\nu$ and $\zeta \in \mathfrak{a}$. $\zeta(0)$ is pure imaginary for all $\zeta \in \mathfrak{a}$.

$\mathfrak{h}_\nu = V \oplus iV$ where V is the real subspace on which the "linear parts" $\tilde{\omega} = \omega - \omega(0)$ of the elements $\omega \in \mathfrak{a}$ are real. One defines

$$\mathfrak{A} = \{\tilde{\omega} \mid V - i\omega(0) : \omega \in \mathfrak{a}\}$$

interpreted as a set of affine functionals $V \to \boldsymbol{R}/\boldsymbol{Z}$. This is the system defined by de Siebenthal.

$\mathfrak{g} = \mathfrak{g}_* \oplus i\mathfrak{g}_*$ where $\mathfrak{g}_*$ is the compact real form of $\mathfrak{g}$. Let s_1 and s_2 be involutive automorphisms of $\mathfrak{g}_*$, σ_1 and σ_2 the extensions of these to anti-involutions of $\mathfrak{g}$. There correspond symmetric subalgebras $\mathfrak{k}_1$, $\mathfrak{k}_2$ of $\mathfrak{g}_*$ and noncompact real forms $\mathfrak{g}_1$, $\mathfrak{g}_2$ of $\mathfrak{g}$.

Let $\mathfrak{m} \subset \mathfrak{g}_*$ be the simultaneous -1 eigenspace of s_1 and s_2. Set $\nu = \sigma_1\sigma_2$ and choose $\mathfrak{h}_\nu$ as in (1.1), but such that $\mathfrak{h}_\nu \cap (\mathfrak{m} \oplus i\mathfrak{m})$ is maxi-

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