PARTIAL DIFFERENTIAL OPERATORS ON $L^p(E^n)$

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Let $P(\xi)$ be a polynomial in the variables $\xi=(\xi_1, \cdots, \xi_n)$. If we replace ξ by $D=(D_1, \cdots, D_n)$, where $D_j=-i\partial/\partial x_j$, $1\leqq j\leqq n$, we obtain a constant-coefficient partial differential operator $P(D)$. Acting on the set C_0^∞ of smooth functions with compact supports in Euclidean n-dimensional space E^n, the operator $P(D)$ is closable in $L^p=L^p(E^n)$ for $1\leqq p\leqq\infty$. The purpose of this note is to describe some of the spectral properties of its closure P_{0p} in L^p.

PROPOSITION 1. *$\sigma(P_{02})$ consists of the closure of the set of values taken on by $P(\xi)$ with ξ real.*

PROPOSITION 2. *A point λ is in $\rho(P_{0p})$ if and only if $1/[P(\xi)-\lambda]$ is a multiplier in L^p* (cf. [1]).

In applying this proposition we shall let $\mu=(\mu_1, \cdots, \mu_n)$ be a multi-index of nonnegative integers. We set $|\mu|=\mu_1+\cdots+\mu_n$ and

$$P^{(\mu)}(\xi)=\partial^{|\mu|}P(\xi)/\partial\xi_1^{\mu_1}\cdots\partial\xi_n^{\mu_n}.$$

With the aid of a theorem of Littman [1] we obtain

THEOREM 3. *Suppose that $1<p<\infty$, and let l be the smallest integer $>n|1/2-1/p|$. Assume that for ξ real*

$$P^{(\mu)}(\xi)/P(\xi)=O(|\xi|^{-a|\mu|})\quad \text{as } |\xi|\to\infty,\quad |\mu|\leqq l, \tag{1}$$

$$1/P(\xi)=O(|\xi|^{-b})\quad \text{as } |\xi|\to\infty, \tag{2}$$

where $b>(1-a)n|1/2-1/p|$. Then $\lambda\in\rho(P_{0p})$ if and only if $P(\xi)\neq\lambda$ for real ξ.

Let $P(\xi)$ and $Q(\xi)$ be polynomials.

PROPOSITION 4. *A necessary and sufficient condition that $D(P_{02})\subseteq D(Q_{02})$ is that*

$$|Q(\xi)|\leqq C(|P(\xi)|+1),\qquad \xi \text{ real}.$$

PROPOSITION 5. *If $\lambda\in\rho(P_{0p})$, then a necessary and sufficient condition that $D(P_{0p})\subseteq D(Q_{0p})$ is that $Q(\xi)/[P(\xi)-\lambda]$ be a multiplier in L^p.*

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