# AN ASYMPTOTIC REPRESENTATION OF THE SAMPLE DISTRIBUTION FUNCTION

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1. Let $X_1, \cdots, X_n$ be independent observations from the uniform distribution on $[0, 1]$. Let $F_n(x)$ = the proportion of the $X_j \leqq x$. We will prove

THEOREM. *There is a random function* $\{G_n(x);\ 0 \leqq x \leqq 1\}$, *with the same distribution as* $\{F_n(x);\ 0 \leqq x \leqq 1\}$ *for each* $n$, *and there is a Brownian motion* $W$, *such that for the Brownian* $B(x) = n^{-1/2}W(nx)$

$$\begin{aligned}(1)\qquad &\sup_{0 \leqq x \leqq 1} \left| n^{1/2}[G_n(x) - x] - [B(x) - xB(1)] \right| \\ &\qquad = O[n^{-1/4}(\log n)^{1/2}(\log \log n)^{1/4}]\end{aligned}$$

*almost surely as* $n \to \infty$.

This theorem is of use in the investigation of the asymptotic behavior of functionals of $\{F_n(x);\ 0 \leqq x \leqq 1\}$, especially functionals dependent on $n$.

2. We construct $G_n(x)$ as follows; let $Y_1, Y_2, \cdots$ be independent exponential variables with mean 1. Let $S(k) = Y_1 + \cdots + Y_k$, $k = 1, 2, \cdots$ and let $S(0) = 0$. Set

$$G_n(x) = k/n \qquad \text{if } S(k)/S(n+1) \leqq x < S(k+1)/S(n+1).$$

This $\{G_n(x);\ 0 \leqq x \leqq 1\}$ has the same distribution as $\{F_n(x);\ 0 \leqq x \leqq 1\}$ for each $n$. We now record a series of lemmas.

LEMMA 1. *There is a Brownian motion* $W$ *such that*

$$(2)\qquad \sup_{1 \leqq k \leqq n} \left| k - S(k) - W(k) \right| = O[n^{1/4}(\log n)^{1/2}(\log \log n)^{1/4}]$$

*almost surely as* $n \to \infty$.

PROOF. This result is deducible from Theorem 1.5 of Strassen [8].

LEMMA 2. *Almost surely as* $n \to \infty$

$$(3)\qquad \sup_{0 \leqq x \leqq 1} \left| S(nG_n(x)) - xS(n+1) \right| = O[n^{1/4}].$$