A METHOD FOR COMPARING UNIVALENT FUNCTIONS

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1. The Loewner representation. Let f be any function in the class S of univalent functions on the unit disk bearing the normalization $f(0)=0, f'(0)=1$. Then it is known [1], [2], that f can be represented as

$$f(z) = \lim_{t\to\infty} e^t h(z, t), \tag{1}$$

locally uniformly in z on $|z|<1$, where $h(z, \cdot)$ is the solution of Loewner's equation in its general form

$$\frac{dh}{dt} = -hp(h, t) \qquad \text{a.e. in } t \text{ on } [0, \infty) \tag{2}$$

with the initial values

$$h(z, 0) = z, \qquad |z| < 1. \tag{3}$$

Here $p(\cdot, t)$ denotes a suitably chosen one-parameter family of holomorphic functions on the unit disk having positive real part and normalized so that $p(0, t)\equiv 1$, whose dependence on t is Lebesgue measurable on $[0, \infty)$ whenever the first variable is held fixed, and the solution of (2) is understood in the Carathéodory sense [3].

Conversely, if $p(\cdot, t)$ denotes any family of functions satisfying the above requirements, then the solution to the foregoing initial-value problem is known to exist and be holomorphic and univalent in z on the unit disk; the limit in (1) is then also known to exist and to determine a function in class S [1], [2].

More generally, one can consider the general solution $h(z, s, t)$ of (2) for $0\leqq s\leqq t$ with the initial values

$$h(z, s, s) = z, \qquad |z| < 1.$$

Then, in place of (1), one has

$$\lim_{t\to\infty} e^t h(z, s, t) = g(z, s)$$

locally uniformly in z for $|z|<1$, where $e^{-s}g(z, s)$ now belongs to S for *all* s in $[0, \infty)$. The function g is absolutely continuous in s and constitutes an integral of Equation (2), for it is easily shown that