# BOUNDARY VALUE PROBLEMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Let $E^n$ denote Euclidean $n$ space with norm $|\cdot|$ and let $C_h$ denote the space of continuous $E^n$ valued functions on $[a-h, a]$, $h>0$, with uniform norm $\|\cdot\|$. For a function $x(t)$ on $[a-h, b]$ and $t\in[a, b]$ let $x_t$ denote the function on $[a-h, a]$ whose value at $\theta$ is $x(t+\theta-a)$. Let $f(t, \Psi)$ be a mapping from $[a, b]\times C_h$ into $E^n$ and let $M$ and $N$ be linear operators from $C_h$ to $C_h$. In this paper we consider special cases of the boundary value problem

$$y' = f(t, y_t), \tag{1}$$

$$My_a + Ny_b = 0, \qquad b > a + h. \tag{2}$$

This is a nonlinear version of a problem posed by Cooke [1]. Other boundary value problems for functional differential equations have been studied recently by Grimm and Schmitt [3], Halanay [4], and Kato [6].

We treat the problem for bounded $f$ and a restricted class of operators by initial value methods, that is, we seek to find an initial function $q\in C_h$ such that a solution of the initial value problem (1) and

$$y(t) = q(t), \qquad a - h \leqq t \leqq a \tag{3}$$

satisfies the boundary condition (2). Some functions $f(t, y_t)$, not bounded, can be treated by approximation techniques and some nonhomogeneous boundary conditions can also be considered.

THEOREM 1. *Let $f(t, \Psi)$ be a continuous bounded function from $[a, b]\times C_h$ into $E^n$ and let $M$ and $N$ be $n\times n$ matrices such that $M+N$ is nonsingular. If $\|(M+N)^{-1}N\|<1$, then there exists a solution of* (1) *and* (2).

METHOD OF PROOF. We give a brief sketch of the method of proof of Theorem 1. The proofs of the other theorems are similar. Assume first that solutions of the initial value problem are unique. Let $T$: $C_h\to C_h$ be defined as follows: for $q\in C_h$, let $Tq=x_b(q)$, that is, $Tq$ is the segment at $b$ of the solution of the initial value problem with

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