AN AMENABLE GROUP WITH A NONSYMMETRIC GROUP ALGEBRA[1]

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Let G be a discrete group, $l_1(G)$ the group algebra of G. Symmetry of $l_1(G)$ has been considered in [1], [3]. Groups containing a free subgroup on two or more generators are the only groups found to have nonsymmetric group algebras, and in each case the groups found to have symmetric algebras are in the family of amenable groups. In this note we present an example of an amenable group with a nonsymmetric group algebra.

LEMMA 1. *Let G be a group generated by a and b such that S, the semigroup generated by a and b, is free and such that $cd^{-1} = dc^{-1}$ for $\{c, d\} = \{a, b\}$. Then $l_1(G)$ is nonsymmetric.*

PROOF. We will show that $l_1(G)$ is not symmetric by showing that the involution is not hermitian. In particular, we will show that $-i \in \mathrm{sp}(x)$ where

$$x = a + ib + a^{-1} - ib^{-1}$$

(we do not distinguish between G and its canonical image in $l_1(G)$). This is accomplished by defining a θ in $m(G)$, the bounded complex valued functions on G, such that $\|\theta\| = 1 = \theta(e)$ and such that

$$\theta^v[(x + ie)g] = 0$$

for each $g \in G$, where $\theta \to \theta^v$ is the mapping of $m(G)$ onto $l_1(G)^*$.

Let $S' = S \cup S^{-1} \cup \{e\}$, and define $\theta(g) = 0$ for $g \in G \sim S'$.

We divide the elements of G into the five disjoint sets; S, S^{-1}, $S_1 = a^{-1}bS \cup \{a^{-1}b\}$, $S_2 = ab^{-1}S^{-1} \cup \{ab^{-1}\}$ and $S_3 = G \sim (S \cup S^{-1} \cup S_1 \cup S_2)$. Let $A = \{a, b, e, a^{-1}, b^{-1}\}$. Direct computations yield $Ag \cap S' \neq \varnothing$ if and only if $g \notin S_3$ or $g = e$.

Note now that if $g \in S_3$ and $g \neq e$ then

$$\begin{aligned}[\text{support } (x + ie)g] \cap S' &= Ag \cap S' \\ &= \varnothing,\end{aligned}$$

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