OVERDETERMINED SYSTEMS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS[1]

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Introduction. The purpose of these lectures is to report on some recent developments in the theory of overdetermined systems of linear partial differential equations. The simplest, and most classical, example of an over-determined system is provided by the equations

$$\frac{\partial f}{\partial x^k} = \phi_k, \qquad k = 1, 2, \cdots, n,$$

where the $\phi_k = \phi_k(x)$ are given (differentiable) functions, $f = f(x)$ is a function to be determined, and $x = (x^1, x^2, \cdots, x^n)$. The system is locally solvable for f if and only if the compatibility conditions

$$\frac{\partial \phi_k}{\partial x^l} = \frac{\partial \phi_l}{\partial x^k}, \qquad k, l = 1, 2, \cdots, n,$$

are satisfied. Write $\phi = \sum_{k=1}^{n} \phi_k dx^k$ and introduce the differential operator d where

$$df = \sum_{k=1}^{n} (\partial f / \partial x^k) dx^k;$$

then the above system can be written in the form $df = \phi$ and the compatibility condition becomes simply $d\phi = 0$ where d here denotes the usual exterior differential operator

$$d\phi = \sum_{k<l} (\partial \phi_l / \partial x^k - \partial \phi_k / \partial x^l) dx^k \wedge dx^l.$$

We are thus led to introduce the following complex (initial portion of the de Rham complex)

$$A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2$$

where A^0 is the sheaf of germs of (differentiable) functions, A^1 and A^2 are the sheaves of germs of differential forms of degrees 1 and 2 respectively. The exactness of the complex expresses the local solvability of the overdetermined system under the required compatibility conditions.

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