STABLE MANIFOLDS FOR HYPERBOLIC SETS

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1. Introduction. We present a version of the "Generalized stable manifold theorem" of Smale [2, p. 781]. Details will appear in the Proceedings of the American Mathematical Society Summer Institute on Global Analysis.

Let M be a finite dimensional Riemannian manifold, $U \subset M$ an open set and $f: U \rightarrow M$ a C^k embedding ($k \in Z_+$). A set $\Lambda \subset U$ is a *hyperbolic set* provided

(1) $f(\Lambda) = \Lambda$;

(2) $T_\Lambda M$ has a splitting $E^s \oplus E^u$ preserved by Df;

(3) there exist numbers $C > 0$ and $\tau < 1$ such that for all $n \in Z_+$,

$$\max\{\|(Df| E^s)^n\|, \|(Df| E^u)^{-n}\|\} \leqq C\tau^n.$$

It is known (J. Mather; see also [1]) that the Riemannian metric on M can be chosen so that $C = 1$; we assume $C = 1$ in what follows. The splitting is unique.

Notation. If X is a metric space, $B_r(x) = \{y \in X | d(y, x) \leqq r\}$. If $\boldsymbol{E}$ is a Banach space, $B\boldsymbol{E} = B_1(0)$. If $E \rightarrow X$ is a Banach bundle, $BE = \bigcup_{x \in X} BE_x$.

A submanifold $W \subset M$ is a *stable manifold through x of size β* if $W \cap B_\beta(x)$ is closed and consists of all $y \in B_\beta(x)$ such that $f^n(y)$ is defined and in $B_\beta f^n(x)$ for all $n \in Z_+$.

An *unstable manifold* is defined to be a stable manifold for f^{-1}. Unstable manifolds are easier to handle in proofs, but stable ones are easier to describe notationally. Hence, we confine ourselves to the stable case.

A C^k *stable manifold system with bundle E* is a family of C^k submanifolds $\{W_x\}_{x \in \Lambda}$ such that

(4) there exists $\beta > 0$ such that each W_x is a stable manifold through x of size β;

(5) E is a vector bundle over Λ, and there is a map $\phi: V \rightarrow M$ of a neighborhood V of the zero section of E such that ϕ maps each $V \cap E_x$ diffeomorphically onto W_x;

(6) ϕ is *fibrewise* C^k in this sense: Let $H: A \times R^q \rightarrow p^{-1}A$ be a trivialization of E over $A \subset \Lambda$ with $H(A \times D^q) \subset V$. Then each map $\theta_x = \phi \circ H | x \times D^q: D^q \rightarrow M$ is C^k, and $\theta: A \rightarrow C^k(D^q, M)$ is continuous.