EXTENSION OF A THEOREM OF CARLESON

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One of the main ingredients in Carleson's solution to the corona problem [2] is the theorem characterizing the measures μ on the open unit disk with the property that $f \in H^p$ implies

$$\int_{|z|<1} |f(z)|^p d\mu(z) < \infty, \qquad 0 < p < \infty.$$

Carleson's proof of this theorem involves a difficult covering argument and the consideration of a certain quadratic form (see also [1]). L. Hörmander later found a proof which appeals to the Marcinkiewicz interpolation theorem and avoids any discussion of quadratic forms. The main difficulty in this approach is to show that a certain sublinear operator is of weak type (1, 1). Here a covering argument reappears which is similar to Carleson's but apparently easier (see [4]).

We wish to point out that Hörmander's argument, with appropriate modifications, actually proves the theorem in the following extended form.

THEOREM. *Let μ be a finite measure on $|z| < 1$, and suppose $0 < p \leqq q < \infty$. Then in order that there exist a constant C such that*

$$\left\{ \int_{|z|<1} |f(z)|^q d\mu(z) \right\}^{1/q} \leqq C\|f\|_p \tag{1}$$

for all $f \in H^p$, it is necessary and sufficient that there be a constant A such that

$$\mu(S) \leqq A h^{q/p} \tag{2}$$

for every set S of the form

$$S = \{re^{i\theta} : 1 - h \leqq r < 1,\ \theta_0 \leqq \theta \leqq \theta_0 + h\}. \tag{3}$$

OUTLINE OF PROOF. A standard argument (factoring out Blaschke products) shows it is enough to consider the case $p = 2$. The necessity of (2) is then proved by choosing $f(z) = (1 - \alpha z)^{-1}$, where $|\alpha| < 1$.

Conversely, let $p = 2$ and suppose (2) holds. Since each $f \in H^2$ is the Poisson integral of its boundary function, it will be sufficient to prove that