A GENERALIZATION OF THE AHLFORS-HEINS THEOREM[1]

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Let D be the complex plane cut along the negative real axis. We are going to consider a function u subharmonic in D. Let $M(r) = \sup_{|z|=r} u(z)$ and $m(r) = \inf_{|z|=r} u(z)$. We also introduce, for $r>0$, $v(r) = \limsup_{z\to -r+i0} u(z)$, $\bar{v}(r) = \limsup_{z\to -r-i0} u(z)$ and $u(-r) = \max(v(r), \bar{v}(r))$. In the whole paper, $z = re^{i\theta}$. Our main result is

THEOREM 1. *Let λ be a number in the interval* $(0, 1)$ *and let u* $(\not\equiv -\infty)$ *be a function subharmonic in D that satisfies*

$$u(-r) - \cos \pi\lambda\, u(r) \leqq 0. \tag{1}$$

Then either $\lim_{r\to\infty} r^{-\lambda} M(r) = \infty$ *or*

(A) *there exists a number α such that*

$$\lim_{r\to\infty} r^{-\lambda} u(re^{i\theta}) = \alpha \cos \lambda\theta, \qquad |\theta| < \pi, \tag{2}$$

except when θ belongs to a set of logarithmic capacity zero.

(B) *Given θ_0, $0<\theta_0<\pi$, there exists an r-set Δ_0 of finite logarithmic length such that* (2) *holds uniformly in* $\{z \mid |\theta| \leqq \theta_0\}$ *when r is restricted to lie outside of Δ_0.*

REMARK. When $1/2<\lambda<1$, condition (1) is interpreted in the following way at points where $u(-r) = \infty$.

$$\limsup_{z\to r}(u(x + iy) + u(-x + iy)) \leqq (1 + \cos \pi\lambda)u(r), \tag{1a}$$

$$\limsup_{z\to r}(u(-x + iy) - \cos \pi\lambda\, u(x + iy)) \leqq 0. \tag{1b}$$

Theorem 1 can be compared to the main result of Kjellberg [6].

THEOREM 2. *Let u be subharmonic in the complex plane and let λ be a number in the interval* $(0, 1)$. *If* $m(r) - \cos \pi\lambda\ M(r) \leqq 0$, *then the (possibly infinite) limit* $\lim_{r\to\infty} r^{-\lambda} M(r)$ *exists.*

In order to clarify the connection between Theorem 1 and the Ahlfors-Heins theorem [1], we also state Theorem 1 in the following equivalent way.

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