ON POLYNOMIALS AND ALMOST-PRIMES

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Communicated by Ivan Niven, July 31, 1968

There exist infinitely many numbers n^2-2 having at most 3 prime factors [1], [3]. We prove here that there exist infinitely many numbers p^2-2 (p prime) having at most 5 prime factors; a similar result with the bound 7 instead of 5 can be found in [5] and, under the Riemann hypothesis, with the bound 5. We use the sieve-method, essentially in the version of Jurkat and Richert as given in [6], and also ideas of Kuhn, de Bruijn, and Bombieri.

Let

$$w(u) := u^{-1} \qquad \text{for } 1 \leqq u \leqq 2,$$
$$(uw(u))' := w(u-1) \qquad \text{for } u \geqq 2,$$
$$D(u) := u \qquad \text{for } 0 \leqq u \leqq 1,$$
$$(u^{-1}D(u))' := -u^{-2}D(u-1) \qquad \text{for } u \geqq 1;$$

here we take the right-hand derivative for integers $u \geqq 0$; let w be continuous at $u=2$ and D be continuous at $u=1$. Define

$$\left.\begin{aligned} \lambda(u) &:= e^{\gamma}u^{-1}(uw(u) - D'(u-1)) \\ \Lambda(u) &:= e^{\gamma}u^{-1}(uw(u) + D'(u-1)) \end{aligned}\right\} \quad (u \geqq 1)$$

where γ is the Euler constant.

Let P be the set of all primes $p \equiv \pm 1 \bmod 8$; $p_0 := 1$; denote by p_j the jth number of P in natural order. Denote by μ the Moebius function and by ϕ the Euler function; let

$$V(n) := \sum_{p^a | n} \sum 1, \qquad Q := \{d : \mu(d) \neq 0 \wedge (p \mid d \Rightarrow p \in P)\},$$

$$f(d) := 2^{-V(d)}\phi(d), \qquad g(d) := f(d) \prod_{p|d} (1 - f(p)^{-1}) \quad (d \in Q),$$

$$P(\rho) := \prod_{1 \leq j \leq \rho} p_j, \qquad R(\rho) := \prod_{1 \leq j \leq \rho} (1 - f(p_j)^{-1}),$$

$$S(x, \rho) := \sum_{1 \leq a \leq x;\, a | P(\rho)} g(a)^{-1}.$$

Using generating functions we find

[1] Supported by National Science Foundation Grant GP 9038.