ADJOINTS, NUMERICAL RANGES, AND SPECTRA OF OPERATORS ON LOCALLY CONVEX SPACES

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1. **Introduction.** The results announced here are of two sorts. First, two important tools from operator theory on Hilbert spaces, the notions of *states* and of *numerical ranges* of operators, are extended to the setting of operator theory on general locally convex spaces. We employ several ideas developed in the work of Lumer [1] for extending these methods to Banach spaces, combining these with the generalizations of classical Banach space methods to the locally convex setting announced by the author in [3].

Second, several related results are announced concerning the approximation of spectra of operators by their numerical ranges. These results are new in the classical settings of Hilbert and Banach spaces as well as in the more general setting. An interesting motivating example on Hilbert spaces, couched in terms of familiar notions, is discussed separately in §2, unencumbered by the technical machinery of later sections.

Full details of proof will appear in two places. A short, less technical discussion of these methods as applied to Hilbert spaces, Banach spaces, B^* algebras, and Banach algebras will appear in [6]. The general theory, with concrete applications, will be included in the monograph [5], along with the results announced in [4], which use these methods to provide a unified theory of the generation of semigroups.

2. **Approximation of spectra in Hilbert spaces.** Let $\mathfrak{H}$ be a complex Hilbert space with a fixed reference inner product $(\ ,\ )_0$. Call another inner product $(\ ,\ )$ *equivalent* to $(\ ,\ )_0$ iff the norms $\|u\| = (u, u)^{1/2}$ and $\|u\|_0 = (u, u)_0^{1/2}$ induce the same topology on $\mathfrak{H}$; we view inner products as "noncanonical" structures associated with an underlying topology, subject to change when convenient.

Then, for any such inner product $(\ ,\ )$ on $\mathfrak{H}$, the *numerical range* is

$$W(T, (\ ,\ )) = \{(Tu, u) \mid u \in \mathfrak{H}, \text{ and } \|u\| = 1\}. \tag{1}$$

It is well known that $W(T, (\ ,\ ))$ is a convex set with compact closure containing the spectrum of T. In fact, if $K = W(T, (\ ,\ ))^-$ (the

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