ADDITIVE CATEGORIES AND A THEOREM OF W. G. LEAVITT[1]

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The purpose of this note is to present a new proof of the following theorem of Leavitt.

THEOREM 1. *Let K be a commutative ring (with 1), and q a positive integer. Then there exists a K-algebra R such that, for positive integers m, n,*

$$R^m \cong R^n \quad \textit{if and only if} \quad m \equiv n \pmod{q}.$$

(R^m denotes the free right R-module on m generators.)

We remark at the outset that as in any case K admits a homomorphism onto a field, an obvious change-of-ring argument shows that it is enough to prove that the theorem is valid in the case where K is itself a field.

Three proofs have already been published, by Leavitt [5], by myself [2], and by Cohn [1]. All three take R to be the K-algebra which is in a fairly obvious sense universal for the isomorphism $R^{q+1} \cong R^1$. To complete the proof, Leavitt shows that $R^m \cong R^n$ only if $m \equiv n \pmod{q}$ by means of a long and involved cancellation argument. As he is interested in rings rather than algebras he presents his proof for the case where K is the field of 2 elements; but it should be remarked that with a little patience his argument can be adapted to the case where K is an arbitrary field. The remaining two proofs, which are considerably shorter and simpler, avoid the cancellation argument with the help of suitable 'trace functions,' but they have the disadvantage that they fail completely unless the identity element of K/qK has additive order exactly q.

The present proof has none of the disadvantages of the earlier proofs. Tackling the problem from an entirely new angle—it makes no use of the universal algebra already mentioned—it employs a quite trivial manipulation of infinite matrices to deduce the theorem from the familiar invariance of dimension of finite-dimensional vector spaces: the underlying idea is strikingly similar to that exploited by Hanf in his solution of an analogous problem for Boolean algebras [4].

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