CONVEXITY PROPERTIES OF NONLINEAR MAXIMAL MONOTONE OPERATORS

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Let X be a real Banach space with dual X^*. A *monotone operator* from X to X^* is by definition a (generally multivalued) mapping T such that

$$\langle x - y, x^* - y^* \rangle \geqq 0 \quad \text{whenever} \quad x^* \in T(x), \quad y^* \in T(y)$$

(where $\langle \cdot , \cdot \rangle$ denotes the pairing between X and X^*). Such an operator is said to be *maximal* if there is no monotone operator T' from X to X^*, other than T itself, such that $T'(x) \supset T(x)$ for every X. The *effective domain* $D(T)$ and *range* $R(T)$ of a monotone operator T are defined by

$$D(T) = \{x \mid T(x) \neq \emptyset\} \subset X,$$
$$R(T) = \bigcup \{T(x) \mid x \in X\} \subset X^*.$$

Minty [9] has shown that, when X is finite-dimensional and T is a maximal monotone operator, the sets $D(T)$ and $R(T)$ are *almost convex*, in the sense that each contains the relative interior of its convex hull. The purpose of this note is to announce some generalizations of Minty's result to infinite-dimensional spaces.

A subset C of X will be called *virtually convex* if, given any relatively (strongly) compact subset K of the convex hull of C and any $\epsilon > 0$, there exists a (strongly) continuous single-valued mapping ϕ from K into C such that $\|\phi(x) - x\| \leqq \epsilon$ for every $x \in K$. It can be shown that, in the finite-dimensional case, C is virtually convex if and only if C is almost convex, so that the following result contains Minty's result as a special case.

THEOREM 1. *Let X be reflexive, and let T be a maximal monotone operator from X to X^*. Then the strong closures of $D(T)$ and $R(T)$ are convex. If in addition X is separable, or if X is an L^p space with $1 < p < \infty$, $D(T)$ and $R(T)$ are virtually convex.*

The proof of Theorem 1, which will appear in [12], is made possible by recent results of Asplund [1], [2] concerning the existence of

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