ON THE MINIMAL PROPERTY OF THE FOURIER PROJECTION

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Let C be the space of real 2π-periodic continuous functions normed with the supremum norm. Let P_n denote the subspace of trigonometric polynomials of degree $\leq n$. It is known [1] that the Fourier projection F of C onto P_n is *minimal*; i.e., if A is a projection of C onto P_n then $\|F\| \leq \|A\|$. We prove that F is the only minimal projection of C onto P_n. The proof is constructed by verifying the assertions listed below. Details will appear elsewhere.

ASSERTION. *If there exists a minimal projection different from F, then there exist minimal projections L and H, different from F such that $\frac{1}{2}L + \frac{1}{2}H = F$.*

The proof of this assertion utilizes Berman's equation,

$$F = \frac{1}{2\pi} \int_{-\pi}^{\pi} T_{-\lambda} A T_{\lambda} d\lambda,$$

which is valid for any projection A of C onto P_n. Here T_λ denotes the shift operator $(T_\lambda f)(x) = f(x+\lambda)$.

ASSERTION. *There is a function $K(x, t)$ of two variables such that*
(i) $K(x, \cdot) \in L^1$ *for each fixed x,*
(ii) $K(\cdot, t) \in P_n$ *for each fixed t, and*
(iii) $(Lf)(x) = \int f(t) K(x, t) dt.$

This is proved by extending A to its second adjoint, and applying the Radon-Nikodym theorem to the functionals $\phi(f) = (A^{**}f)(x)$.

Let D_n denote the Dirichlet kernel. The next assertion follows from an examination of the roots of K where K is considered as a function of x.

ASSERTION. *There is a function $g \in L^1$ such that $0 \leq g \leq 2$, and $K(x, t) = g(t) D_n(x-t)$.*

ASSERTION. (i) $(1-g) \perp P_{2n}$ *and* (ii) $(1-g)* |D_n| = 0$ *where * denotes convolution.*

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