# THE FUNDAMENTAL LEMMA OF COMPLEXITY FOR ARBITRARY FINITE SEMIGROUPS[1]

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**1. Statement of the results and some corollaries.** All semigroups considered are of finite order. In the recent paper [3] and in the recent book [2] the complexity of a semigroup was defined and definitive results were obtained for determining the complexity of a semigroup which was the union of groups. Herein we state generalizations, valid for arbitrary finite semigroups, of those previous results. All undefined notation is explained in [2].

We first recall the definition of complexity. See also [2] or [3]. One semigroup, $S_1$, is said to *divide* another semigroup, $S_2$, if and only if $S_1$ is a homomorphic image of a subsemigroup $S \leqq S_2$. If $S$ is a semigroup, $\text{Endo}(S)$ denotes the semigroup of endomorphisms of $S$ under composition. If $S_1$ and $S_2$ are semigroups and $Y$ is a homomorphism of $S_1$ into $\text{Endo}(S_2)$, *the semidirect product of* $S_2$ *by* $S_1$ *with connecting homomorphism* $Y$, denoted by $S_2 \times_Y S_1$, is the semigroup with elements $S_2 \times S_1$ and product defined by $(s_2, s_1) \cdot (s_2', s_1') = (s_2 \cdot Y(s_1)(s_2'), s_1 \cdot s_1')$.

We can construct new semigroups from old ones by taking semidirect products and then divisors. $S_n \times_{Y_{n-1}} \cdots \times_{Y_2} S_2 \times_{Y_1} S_1$ denotes $(\cdots (S_n \times_{Y_{n-1}} S_{n-1}) \times_{Y_{n-2}} S_{n-2}) \cdots \times_{Y_1} S_1)$ where $Y_{n-2}$ is a homomorphism of $S_{n-2}$ into $\text{Endo}(S_n \times_{Y_{n-1}} S_{n-1})$, etc. We say $S$ is a *combinatorial* semigroup if and only if the subsemigroups of $S$ which are groups are singletons. The main theorem of [1] (see also [2, Chapter 5]) implies that for each semigroup $S$ there exist semigroups $S_n, \cdots, S_1$ and connecting homomorphisms $Y_{n-1}, \cdots, Y_1$ so that

$$S \text{ divides } S_n \times_{Y_{n-1}} \cdots \times_{Y_1} S_1 \tag{1.1}$$

and $S_k$ is either a simple nontrivial group dividing $S$ or $S_k$ is a combinatorial semigroup, for $k = 1, \cdots, n$.

$\#_G(S)$, the (group) complexity of $S$, is by definition the smallest nonnegative integer $n$ such that

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