ON THE APPROXIMATION BY C-POLYNOMIALS[1]

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1. Introduction. Throughout this note C denotes the unit circle and D its interior. It is the object of this note to give a simple unified treatment to the problem of approximation of a zero free holomorphic function in D (uniformly on compact sets of D) and the problem of bounded approximation of a zero free bounded holomorphic function in D by C-polynomials; i.e. polynomials whose zeros lie on C.

It is known [1], [4] that such approximations are possible in regions whose boundaries satisfy certain smoothness conditions, but the methods used in [1] and [4] yield different approximating sequences. In particular our Main Theorem implies Theorem 1 of [4] for a disk and the Main Theorem of [1]. The proof of the main result is followed by an application of our method to the problem of C-continuation of polynomials [2], [3].

MAIN THEOREM. *Let $f(z)=1+c_1z+c_2z^2+\cdots$, be a zero free holomorphic function in D. Then there exists a sequence of C-polynomials assuming the value one at $z=0$ which converges to $f(z)$ uniformly on every compact subset of D. If in addition the function $f(z)$ is bounded in D then the sequence converges to $f(z)$ boundedly.*

2. Proof of the main theorem. The following lemma is easily verified by observing some simple properties of the linear transformation $(1-z\alpha)(z-\bar{\alpha})^{-1}$ and applying Rouché's Theorem.

LEMMA. *If $P(z)$ is a polynomial of degree m which does not vanish in D then the zeros of the polynomial $P(z)+z^pP^*(z)$, where $P^*(z)=z^m\overline{P}(z^{-1})$ all lie on C for $p=0, 1, 2, \cdots$. Furthermore $|P^*(z)|\leqq|P(z)|$ for $|z|\leqq 1$.*

Let $s_n(z)=1+c_1z+c_2z^2+\cdots+c_nz^n$, and let r_k $(k=1, 2, \cdots)$ be any sequence of positive numbers strictly increasing to one. There exists a strictly increasing sequence of positive integers n_k such that $s_{n_k}(z)\neq 0$ and

$$|s_{n_k}(z)-f(z)| < 1/k \qquad \text{for } k=1, 2, \cdots$$

and for $|z|<r_k$. Define $t_{n_k}(z)=s_{n_k}(r_kz)$ and $P_{n_k}(z)=t_{n_k}(z)+z^{n_k}t^*_{n_k}(z)$. Since $t_{n_k}(z)$ does not vanish in D it follows by the lemma that the

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