$O(h^{2n+2-l})$ BOUNDS ON SOME SPLINE INTERPOLATION ERRORS[1]

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For four or five years it had been felt that C^{2n}, $2n+1$-degree polynominal spline interpolation of a sufficiently smooth function at equally spaced joints (h apart) yielded $O(h^{2n+2-l})$ accuracy in approximating its lth derivative. This was recently shown to be true for periodic boundary conditions on the spline, s, which interpolates $f\in C^{2n+2}(-\infty, \infty)$, f periodic with period 1 ([**1**, p. 151]; [**2**, Theorem 4] as improved by [**3**, last paragraph]). It is shown here that the errors then are the same (up to a higher order term) as the errors associated with local two-point $2n+1$-degree polynomial interpolation, H, of f and its first n odd derivatives at the joints (Theorem 1). The first term in the asymptotic expansion of $\|f^{(l)}-s^{(l)}\|_\infty$ is also derived; it is quite local in character. Theorem 2 states that the same results hold for the spline interpolating $f\in C^{2n+2}[0, 1]$ which matches f and its first n odd derivatives at 0 and 1 as well.

Complete proofs and further references are given in [**3**]. The somewhat less satisfactory situation for another boundary condition (first n *even* derivatives) is also discussed there together with results for other norms and for rougher functions, f. The emphasis in [**3**] is on strict, rather than asymptotic, bounds.

The foundations of the proof arose from consideration of some cubic spline interpolations at arbitrarily spaced joints. Here the techniques yielded some error bounds better than any yet published. They showed further that for $f\in C^4[0, 1]$, $f-s$ is locally bounded by $O(h_i^2h_M^2)$, where h_i is the local mesh size and h_M is the maximum mesh size [**3**].

The typical proof of $O(h^{2n+2-l})$ errors determines first a bound on a high-order derivative of $(f-s)(x)$; then obtains rough bounds on lower-order derivatives by observing that they all have zeroes reasonably nearby. The technique used here, however, is to write $f-s$ as $f-H$ plus the piecewise polynomial $H-s$. $(f^{(l)}-H^{(l)})(x)$ is computed by classical Green's functions arguments. The main lemmas then bound $H^{(l)}-s^{(l)}$ by $O(h^{2n+2-l}\omega(f^{(2n+2)}, h))$. One thus concludes (with the notation $h\equiv 1/N$, $g_i\equiv g(ih)$ for any function g,

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