DECIDABILITY OF SECOND-ORDER THEORIES AND AUTOMATA ON INFINITE TREES[1]

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1. Introduction. In this note we announce the solvability of the decision problem of the (monadic) second-order theory of two successor functions (S2S). This answers a question raised by Büchi [1].

The above decidability result turns out to be very powerful in that many difficult, often seemingly unrelated, decision problems are reducible to it. Thus we are able to deduce: the decidability of the first-order theory of the lattice of closed subsets of the real line (in answer to Grzegorczyk [6]); the decidability of the second-order theory of countable linearly ordered sets; decidability of theory of countable Boolean algebras with quantification permitted over ideals; and many other results. All the decidability procedures obtained here are elementary recursive in the sense of Kalmar. Due to the fact that we use reductions to a second-order theory, our decidability proofs are very direct. Through appropriate interpretations, the set variables of S2S allow us to talk about all structures in a certain class.

The method of solution involves the development of a theory of finite automata operating on infinite trees. Complete details will be published elsewhere.

1. Theory of n successor functions. Let $T=\{0, 1\}^*$ be the set of all finite words on $\{0, 1\}$. The functions $r_0(x)=x0$, $r_1(x)=x1$, $x\in T$, are called the *successor functions*. On T define the relation $x\leqq y \equiv \exists z[y=xz]$; and the lexicographic total ordering $x\preceq y\equiv x\leqq y \vee \exists z\, \exists u\, \exists v[x=z0u\wedge y=z1v]$.

Let Λ denote the empty sequence. A *path* π of T is a subset $\pi\subset T$ such that (1) $\Lambda\in\pi$; (2) for each $x\in\pi$, either $x0\in\pi$ or $x1\in\pi$; (3) for each $\Lambda\neq x\in\pi$, the predecessor node y of x is in π.

For $\mathfrak{M}$ a structure and L a formal language, $\mathrm{Th}(\mathfrak{M}, L)$ will denote the theory of $\mathfrak{M}$ in the language L. If $\mathcal{K}$ is a class of similar structures, then $\mathrm{Th}(\mathcal{K}, L)=\bigcap_{\mathfrak{M}\in\mathcal{K}} \mathrm{Th}(\mathfrak{M}, L)$. If L is (monadic) second-order, then we denote $\mathrm{Th}(\mathfrak{M}, L)$ by $\mathrm{Th}_2(\mathfrak{M})$. If L' is second-order and the set variables are restricted to range over finite subsets of the domain, then $\mathrm{Th}(\mathfrak{M}, L')$ is called the *weak second-order theory* of $\mathfrak{M}$.

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