MAXIMAL IDEALS IN TENSOR PRODUCTS OF BANACH ALGEBRAS

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In this note we characterize the maximal left ideals in certain tensor products of complex Banach algebras. Although our methods are different, the results presented here resemble results of Gelbaum [1], [2] that characterize the maximal two-sided ideals in the greatest cross norm tensor product of Banach algebras.

Let A be a commutative Banach algebra with identity 1, and let B be an arbitrary Banach algebra with identity e. It follows from the universal property [3, p. 181] of the algebraic tensor product $A \otimes B$ that if M is a maximal ideal of A, then M induces a homomorphism h_M of $A \otimes B$ onto B by the formula

$$h_M \sum a_i \otimes b_i = \sum a_i(M)b_i.$$

In all that follows we denote by $\mathfrak{A}$ the completion of $A \otimes B$ in some cross norm such that each homomorphism h_M is bounded, hence has a unique extension to $\mathfrak{A}$. The greatest cross norm has this property. When A is semisimple, another cross norm with this property is the sup norm

$$\left\| \sum a_i \otimes b_i \right\| = \sup \left\| \sum a_i(M)b_i \right\|.$$

With these conventions understood the main result can be stated as follows.

THEOREM. *A subset $\mathfrak{L}$ of $\mathfrak{A}$ is a maximal left ideal if and only if A contains a maximal ideal M and B contains a maximal left ideal L such that* $\mathfrak{L} = h_M^{-1}(L)$.

PROOF. To prove the sufficiency, let us suppose $\mathfrak{L}$ has the desired form and that $\mathfrak{L}'$ is a left ideal which properly contains $\mathfrak{L}$. Because h_M is onto B, $h_M(\mathfrak{L}')$ is a left ideal which properly contains L. Since L is maximal in B, it follows that there is an element x in $\mathfrak{L}'$ such that $h_M(x) = e$. Now $x - 1 \otimes e$ belongs to the kernel of h_M which is contained in $\mathfrak{L}'$, thus $1 \otimes e$ belongs to $\mathfrak{L}'$ and so $\mathfrak{L}' = \mathfrak{A}$, proving that $\mathfrak{L}$ is maximal.

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