INVARIANT SUBSPACES AND WEAKLY CLOSED ALGEBRAS

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1. **Introduction.** One of the reasons that reducing subspaces of operators on a Hilbert space are more easily studied than invariant subspaces is the relation between reducing subspaces and von Neumann algebras. If $\mathfrak{A}$ is a von Neumann algebra containing the identity operator and B is an operator such that every reducing subspace of $\mathfrak{A}$ reduces B, then B is in $\mathfrak{A}$.

If $\mathfrak{A}$ is any algebra of operators and if $\mathfrak{A}'$ is the algebra of all operators which leave invariant every invariant subspace of $\mathfrak{A}$, then clearly $\mathfrak{A}\subseteq\mathfrak{A}'$. Professor P. R. Halmos has suggested the following definition.

DEFINITION. An algebra $\mathfrak{A}$ of operators on a Hilbert space is *reflexive* if whenever every subspace invariant under $\mathfrak{A}$ is invariant under B, then B is in $\mathfrak{A}$.

Examples of reflexive algebras are easily constructed: if $\mathfrak{F}$ is any family of subspaces of a Hilbert space $\mathfrak{H}$ and if $\mathfrak{A}$ is the algebra of all operators which leave every subspace in $\mathfrak{F}$ invariant, then $\mathfrak{A}$ is reflexive. Clearly, every reflexive algebra is weakly closed and contains the identity operator. Also every von Neumann algebra containing the identity is reflexive.

In this paper we consider two sufficient conditions that a weakly closed algebra be reflexive. Our work is based upon elegant techniques introduced in recent papers of D. E. Sarason [**4**] and W. B. Arveson [**1**]. The main results of Sarason and of Arveson may be stated as follows.

SARASON'S THEOREM. *If $\mathfrak{A}$ is a weakly closed algebra (containing the identity) of commuting normal operators, then $\mathfrak{A}$ is reflexive.*

ARVESON'S THEOREM. *If $\mathfrak{A}$ is a weakly closed algebra which contains a maximal abelian von Neumann algebra and which has no invariant subspaces except $\{0\}$ and $\mathfrak{H}$, then $\mathfrak{A}$ is reflexive (i.e. $\mathfrak{A}$ is the algebra of all operators on $\mathfrak{H}$).*

Below we give two theorems that are similar to the above. Our main result (Theorem 1) is a generalization of Arveson's Theorem.

2. **Statement of results.**

THEOREM 1. *A weakly closed algebra which contains a maximal abelian von Neumann algebra and whose lattice of invariant subspaces is totally ordered is reflexive.*