EMBEDDINGS WITH CODIMENSION TWO OF SPHERES IN SPHERES AND H-COBORDISMS OF $S^1 \times S^3$

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1. Embeddings with codimension two of spheres in spheres. Let S^i denote the standard i-sphere with the usual smooth structure, and write $S^i \subseteq S^{i+j}$ for the standard inclusion of S^i into S^{i+j} on the first $(i+1)$ coordinates.

THEOREM 1.1. *Let $\phi: S^3 \to S^5$ be a smooth embedding. Then the following are equivalent:*

(1) *$S^5 - \phi S^3$ has the homotopy type of a circle; and*

(2) *there is a diffeomorphism $f: S^5 \to S^5$, isotopic to the identity, such that $f\phi$ is the standard inclusion of S^3 in S^5.*

That (2) implies (1) is easy. Note that (2) is stronger than the usual unknotting criterion that there be an orientation preserving diffeomorphism of S^5 throwing ϕS^3 onto the image of S^3 under the standard inclusion $S^3 \subseteq S^5$. J. Levine [1] has proven theorems analogous to 1.1 for embeddings of S^n in S^{n+2}, $n \geqq 4$, but with (2) replaced by this weaker criterion. This difference is explained by Theorem 1.3 below.

To prove Theorem 1.1, we use the next result.

THEOREM 1.2. *Let M^5 be a smooth manifold of the same homotopy type as $S^1 \times S^4$. Then M is diffeomorphic to $S^1 \times S^4$.*

This theorem is proved in the author's thesis and will appear in print with a proof elsewhere. Assuming 1.2, let ϕ be as in the statement of 1.1, and suppose that ϕ satisfies condition (1). Let $\eta: S^3 \times D^2 \to S^5$ be a smooth embedding such that $\eta(x, 0) = \phi(x)$; η exists because $\pi_2(\mathrm{SO}(2)) = 0$. Let W be the closure of the complement of Image η. Let M be obtained by surgery using η; i.e. $M = W \cup_\eta D^4 \times S^1$, the disjoint union with (x, y) and $\eta(x, y)$ identified for (x, y) in $S^3 \times S^1$. There is a homotopy equivalence $\alpha: D^4 \times S^1 \to W$ such that $\alpha / S^3 \times S^1 = \eta / S^3 \times S^1$. This is not hard to see using the fact that W has the homotopy type of a circle. Hence there is a map

$$\alpha \cup id: \ D^4 \times S^1 \cup_{id} D^4 \times S^1 \to W \cup_\eta D^4 \times S^1 = M.$$

It is easy to see that this map is a homotopy equivalence. Moreover, $D^4 \times S^1 \cup_{id} D^4 \times S^1 = S^4 \times S^1$. If x is in D^4 and y in S^1, let $(x, y)_2$ denote