CONNECTION PROBLEMS FOR ASYMPTOTIC SERIES[1]

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1. Connection problems and asymptotic power series. An analytic function is completely determined by the coefficients of its power series about one regular point, e.g. by

$$f(z) = \sum_{r=0}^{\infty} g_r z^r. \tag{1.1}$$

The principle of analytic continuation makes it possible to calculate effectively the corresponding convergent power series about all points where a holomorphic continuation exists. However, the nature of the function near its singularities cannot be so readily deduced from the series (1.1).

Often series expansions about such singular points do exist, and sometimes it is possible to calculate them explicitly from the coefficients of the convergent expansions about a regular point. These expansions may even be power series. Nevertheless, they differ from the familiar convergent Taylor series in several decisive respects. The most important new feature is that *they represent the function f only in an asymptotic sense.* To explain this concept, let us assume, for simplicity, that the singularity occurs at $z=\infty$. To say that the function f is asymptotically represented by the series $\sum_{r=0}^{\infty} c_r z^{-r}$, in symbols $f(z) \sim \sum_{r=0}^{\infty} c_r z^{-r}$, as $z \to \infty$, means that, for all N, the error committed in replacing $f(z)$ by the sum of the first N terms of the power series is $O(z^{-N})$, as $z \to \infty$. Such a series may well be divergent, in fact, it usually is. If so, another important feature enters the picture: *A divergent asymptotic series for an analytic function at an isolated singularity never represents the function in a full deleted neighborhood, but only in certain sectors.*

There exists a substantial body of theory for the "connection problem" just described, namely the problem of finding asymptotic expansions about a singular point from a given convergent expansion for the same function about a regular point. I shall not say much

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