# ON THE DERIVATIVE OF A SEMIGROUP[1]

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Let $I$ be a countably infinite set, and $P = \{P(t, i, j)\}$ a standard semigroup on $I$: that is, $P(t)$ is a stochastic matrix, $P(t+s) = P(t)P(s)$, and

$$\lim_{t \to 0} P(t, i, i) = P(0, i, i) = 1 \quad \text{for all } i \in I.$$

As is well known, $Q = P'(0)$ exists, although $q(i) = Q(i, i)$ may be infinite for some or all $i$. When $q(i) < \infty$, the numbers $q(i)$ and $Q(i, j)/q(i)$ have interesting known probabilistic interpretations, although the meaning of $Q(i, j)$ itself is a little obscure. The object of this note is to "explain" $Q(i, j)$ in a way which does not depend on $q(i)$, finite or infinite.

To state the explanation, give $I$ the discrete topology, and let $I \cup \{\phi\}$ be the one-point compactification. On a suitable probability triple, say $(\Omega, \mathfrak{F}, P_k)$, construct an $I \cup \{\phi\}$-valued process $X$, which is Markov with stationary transitions $P$, starts from $k \in I$, and has smooth sample functions.

More formally, for $0 = t_0 < t_1 < \cdots < t_n$ and $i_0 = k$ and $i_1, \cdots, i_n$ in $I$,

$$P_k\{X(t_m) = i_m \text{ for } m = 0, \cdots, n\} = \prod_{m=0}^{n-1} P(t_{m+1} - t_m, i_m, i_{m+1}).$$

Moreover, for each $\omega$ and all $t > 0$, as rational $r$ increases to $t$, the (generalized) sequence $X(r, \omega)$ has at most one limiting value in $I$. (This does not exclude the possibility of having $\phi$ as a limiting value or even converging to $\phi$.) Finally, for each $\omega$ and all $t \geqq 0$, as rational $r$ decreases to $t$, there are only two possibilities: either $X(t, \omega) = \phi$ and $X(r, \omega)$ tends to $\phi$; or $X(t, \omega) \in I$ and $X(r, \omega)$ has precisely one limiting value in $I$, namely $X(t, \omega)$.

As is known, such a construction is always possible.

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