PLANARITY IN ALGEBRAIC SYSTEMS

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Planarity was introduced into algebra by Marshall Hall in his well-known coordinatization of a projective plane by a planar ternary ring [4]. In [6], J. L. Zemmer defines a near-field to be planar when the equation $ax=bx+c$ has a unique solution for $a\neq b$. In our investigation of planarity, we discovered that if $(N, +, \cdot)$ is a near-ring satisfying the above equational property, then $(N, +, \cdot)$ is a near-field. (This was conjectured by both D. R. Hughes and J. L. Zemmer in private communications.) We present some extensions of this result together with geometric interpretations of "planar" near-rings.

Definitions and notations. By a *left distributive system* is meant a triple $(N, +, \cdot)$ such that multiplication $\cdot$ is left distributive over addition $+$. Elements $a, b\in N$ are called *left equivalent multipliers*, denoted by $a\equiv_m b$ *iff* $ax=bx$ for all $x\in N$. The relation $\equiv_m$ is *discrete* when $a\equiv_m b$ implies $a=b$. A left distributive system is said to possess the *planar property* if the equation $ax=bx+c$ has a unique solution for $a\not\equiv_m b$.

DEFINITION. A left distributive system $(N, +, \cdot)$ with planar property is a *planar system* if

(1) in $(N, +)$ the right cancellation law is valid;
(2) in $(N, +)$ there is an identity 0;
(3) $(N, \cdot)$ is a semi-group;
(4) there are at least three points in N, no two of which are left equivalent multipliers.

A planar system is *integral* if 0 is the only left zero divisor.

Main results. Let $(N, +, \cdot)$ be an integral planar system. Then $0\cdot x=x\cdot 0=0$ for all $x\in N$. Let 1_a be the solution to the equation $a\cdot x=a$, $a\neq 0$, and $B_a=\{x\in N^* \mid x\cdot 1_a=x\}$, where N^* denotes the nonzero elements of N. We have the following

THEOREM 1. *Let* $(N, +, \cdot)$ *be an integral planar system. Then*
(i) *each* $(B_a, \cdot)$ *is a group with identity* 1_a;
(ii) *the family* $\{B_a\}_{a\in N^*}$ *is pairwise disjoint*;
(iii) $N^*=\bigcup_{a\in N^*} B_a$;

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