PROOF OF A CONJECTURE OF HELSON[1]

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Let m_n denote the Haar measure of the torus T^n, the distinguished boundary of the unit polydisc U^n in the space of n complex variables. If f is holomorphic in U^n, define

$$f^*(z) = \lim_{r\to 1} f(rz) \tag{1}$$

for those $z \in T^n$ for which this radial limit exists. Here $z = (z_1, \cdots, z_n)$, $rz = (rz_1, \cdots, rz_n)$. The various H^p-norms in U^n, for $0 < p < \infty$, $n = 1, 2, 3, \cdots$, are defined by

$$\|f\|_{p,n} = \sup_{0<r<1} \left\{ \int_{T^n} |f(rz)|^p dm_n(z) \right\}^{1/p}. \tag{2}$$

As in one variable, the inequality

$$\log |f(0)| \leqq \int_{T^n} \log |f^*(z)| \, dm_n(z) \tag{3}$$

holds for every $f \in H^p(U^n)$. Define

$$\Delta(f) = \int_{T^n} \log |f^*(z)| \, dm_n(z) - \log |f(0)| . \tag{4}$$

For $f \in H^2(U^n)$, let $S(f)$ denote the H^2-closure of the set of all products Pf, where P ranges over the polynomials in n variables; $S(f)$ is the *invariant subspace of* $H^2(U^n)$ *generated by* f.

A very well-known theorem of Beurling states (in one variable) that

$$S(f) = H^2(U) \quad \textit{if and only if} \quad \Delta(f) = 0. \tag{5}$$

One of these implications holds equally well for several variables, as has been known for quite some time to Helson and Lowdenslager: *If* $f \in H^2(U^n)$ *and* $S(f) = H^2(U^n)$, *then* $\Delta(f) = 0$. Here is a sketch of a simple proof: (i) $\Delta(Pf) = \Delta(P) + \Delta(f) \geqq \Delta(f)$ for all P. (ii) Δ is an upper semicontinuous function on $H^2(U^n)$. (iii) Therefore $\Delta(g) \geqq \Delta(f)$ for every $g \in S(f)$.

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