A COMBINATION OF MONTE CARLO AND CLASSICAL METHODS FOR EVALUATING MULTIPLE INTEGRALS

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1. Stochastic quadrature formulas. In the simplest "Monte Carlo" scheme for numerically approximating the integral

$$I = \int_{G_s} f(\mathbf{x})d\mathbf{x} \tag{1}$$

(G_s is the closed unit cube in E^s), N points $\mathbf{x}_1, \cdots, \mathbf{x}_N$ are chosen at random in G_s and the quantity

$$J_0 = \frac{1}{N}\sum_{i=1}^{N} f(\mathbf{x}_i)$$

is taken as an estimate of I. The error analysis is probabilistic. Regarding the $\mathbf{x}_i$ as (pairwise) independent random variables uniformly distributed on G_s, J_0 is a random variable with mean I; the amount by which it is apt to differ from I is estimated in terms of its standard deviation $\sigma(J_0)$. In general (for $f \in L^2(G_s)$),

$$\sigma(J_0) = C_0(f)N^{-1/2};$$

and it is usual to consider 3σ (or even 2σ) as a reliable upper bound on $|J-I|$.

Let D_s^n denote the set of real functions f such that

$$\frac{\partial^{n_1+\cdots+n_s}}{(\partial x^1)^{n_1}\cdots(\partial x^s)^{n_s}} f(x^1, x^2, \cdots, x^s)$$

is continuous on G_s whenever $n_1, n_2, \cdots, n_s \leqq n$. N. S. Bahvalov [1], in a study of lower bounds on quadrature errors showed that for the class D_s^n the error of any nonrandom (e.g. Newton-Cotes, Gaussian) quadrature method is $\Omega(N^{-n/s})$;[1] for random methods the best he could show was $\sigma = \Omega(N^{-(n/s+1/2)})$ and he showed that for the set of periodic functions in D_s^n there in fact exist methods for which $\sigma = O(N^{-(n/s+1/2)})$.

In this note I shall give a general description of a class of formulas which combine the Monte Carlo and classical approaches to get

[1] Hardy's notation: $f = \Omega(g)$ iff $g = O(f)$.