GERŠGORIN THEOREMS BY HOUSEHOLDER'S PROOF

BY J. L. BRENNER

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0. **The method.** Given an $m \times m$ matrix $A = [a_{ij}]$ of complex numbers, S. Geršgorin [4] proved that every proper value λ lies in the union of the m disks D_i, where $D_i \equiv \{\lambda \mid |\lambda - a_{ii}| < R_i,\ R_i = \sum_{j \neq i} |a_{ij}|\}$. Generalizations of this theorem have appeared in several papers, see for example [1], [3], [5], [7], [8], and a convenient summary in [6]. The theorem is derivable from the following (older) result, if we set $B = A - \lambda I$.

THEOREM 1. *Let $B = [b_{ij}]$ be a matrix of complex numbers. If B is not invertible, then for some i we must have* $|b_{ii}| \leqq \sum_{j \neq i} |b_{ij}| = R_i$.

COROLLARY. $\forall_i \{ |b_{ii}| > R_i \} \Rightarrow B$ *is invertible.*

This is the contrapositive of Theorem 1. To prove Theorem 1, find $x = \{x_1, x_2, \cdots, x_n\}$ so that $Bx = 0$; choose i so that $x_i \neq 0$ and $\forall_j \{ |x_i| \geqq |x_j| \}$. Then $|b_{ii}| \leqq \sum |b_{ij}| \cdot |x_j / x_i| \leqq R_i$.

Householder [5, p. 66] looks at the theorem from a different point of view. He writes $B = D - C$, where D is the diagonal part of B, i.e. $D = [d_{ij}]$, $d_{ij} = \delta^i_j \cdot b_{ij}$, and C has zero diagonal. If $\forall_i \{b_{ii} \neq 0\}$, then $B = D(I - D^{-1}C)$. The condition $\|D^{-1}C\| < 1$ guarantees that B be invertible. The corollary follows on applying this condition and using the row-sum norm.

1. **A new result.** In the preceding paragraph, a known result was recovered by Householder's method. This does not demonstrate the full power of the method. In this section, we obtain a new result by the same method. (This result can be obtained also by other methods; see [2].)

DEFINITION. The notation

$$B\begin{pmatrix} 1 \cdots n \\ 1 \cdots n \end{pmatrix}$$

means the minor matrix obtained from the large matrix B by retaining only rows $1 \cdots n$ and columns $1 \cdots n$. The notation

$$B\begin{pmatrix} 1 \cdots n \\ \{1 \cdots n\} \backslash t, j \end{pmatrix}$$