ON THE REPRESENTATION THEOREM OF SCATTERING

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1. Introduction. We shall show that Theorem 1.1 on the Outgoing Translational Representation of a unitary group due to Lax and Phillips [4, Theorem 1], [5, Corollary 3.2, p. 51] follows readily from the Wold Decomposition for isometric flows obtained by the writer [7] in 1962. Our treatment will also reveal the "accessory" Hilbert space, give an explicit integral representation for the isometry occurring in the conclusion of the theorem (W and Σ in 1.1), and place in perspective the Spectral Representation due to Sinai [9], (cf. [5, Theorem 3.1, p. 50]).

1.1. THEOREM (LAX-PHILLIPS). *Let $\mathcal{H}_0$ be an outgoing subspace of a (complex) Hilbert space $\mathcal{H}$ relative to the strongly continuous group $(U_t, -\infty < t < \infty)$ of unitary operators on $\mathcal{H}$ onto $\mathcal{H}$; i.e. let*

$$\text{(i)}\quad \forall t > 0, \quad U_t(\mathcal{H}_0) \subseteq \mathcal{H}_0,$$

$$\text{(ii)}\quad \bigcap_{t>0} U_t(\mathcal{H}_0) = \{0\}, \qquad \text{(iii)}\quad \text{cls.} \bigcup_{t<0} U_t(\mathcal{H}_0) = \mathcal{H}.$$

Then there exists a Hilbert space W and a unitary operator Σ on $L_2((-\infty, \infty); W)$ onto $\mathcal{H}$ such that[2]

$$\Sigma\{L_2([0, \infty); W)\} = \mathcal{H}_0; \qquad \Sigma \circ \tau_{-t} \circ \Sigma^{-1} = U_t, \quad -\infty < t < \infty,$$

where τ_t is translation through t, i.e. $(\tau_t x)(s) = x(t+s)$.

We shall first indicate our proof in the discrete case. Replace the real t in 1.1 by the integer k and U_t by U^k. The hypothesis 1.1(i) then tells us that $(\text{Rstr.}_{\mathcal{H}_0} U^k, k \geqq 0)$ is a semigroup of isometries on $\mathcal{H}_0$ to $\mathcal{H}_0$. Now for any discrete semigroup $(V^k, k \geqq 0)$ of isometries on a Hilbert space $\mathcal{H}$ to $\mathcal{H}$ we have the Wold decomposition, (cf. Halmos [2, Lemma 1] or [7, (1.1)]):

$$(1.2)\qquad \begin{aligned} \mathcal{H} &= \mathcal{H}_\infty + \sum_{k=0}^{\infty} V^k(W), & \mathcal{H}_\infty &\perp \sum_{k=0}^{\infty} V^k(W) \\ \mathcal{H}_\infty &=_{\mathrm{d}} \bigcap_{k \geq 0} V^k(\mathcal{H}), & W &=_{\mathrm{d}} \{V(\mathcal{H})\}^{\perp}. \end{aligned}$$

Hence in our case, cf. 1.1(ii),

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[2] In this paper $L_2([a, b]; W)$ shall denote the set of all functions in $L_2((-\infty, \infty); W)$ essentially supported on $[a, b]$. $\text{Rstr.}_S F$ means: restriction of F to S. For $=_{\mathrm{d}}$ read "equals by definition."