GENERATORS AND RELATIONS FOR CERTAIN SPECIAL LINEAR GROUPS

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Several years ago I calculated presentations for several of the groups SL(2, R) where R is the ring of integers of a quadratic imaginary number field $K=\boldsymbol{Q}((-m)^{1/2})$. The method used was extremely tedious and was never published. Recently, while checking these calculations, I discovered a much simpler approach to the problem which I will outline here. The interest in these calculations is considerably increased by recent results of Serre [6]. He considers the congruence subgroup problem for the groups SL(2, R) where R is the ring of integers $\mathfrak{O}$ of an algebraic number field (and, more generally for $R=\mathfrak{O}[a^{-1}]$ where $a\in\mathfrak{O}$). He obtains the expected results [1], [5] whenever R has a unit of infinite order. Thus the only exceptions are $R=\boldsymbol{Z}$ and the case which I will consider here. Serre has also shown that all of these cases are true exceptions. The case $R=\boldsymbol{Z}$ is, of course, well known. Hopefully, the calculations outlined here will throw some light on the remaining cases. At present, I have only carried out the calculations for fields K with discriminants D between -1 and -24. The length of the calculation increases rapidly with $|D|$ but the calculation could easily be extended to arbitrarily large values of $|D|$ by machine computation. This has not been done at the present time. Full details of the calculations will be published elsewhere. I would like to thank H. Bass for communicating Serre's results to me.

1. Transformation groups. The original calculation depended on a theorem of Macbeath [4]. However, this leads to an excessively large number of generators and relations and so to the long and tedious process of simplifying the presentation. The main simplification results from a generalization of Macbeath's theorem to non-simply-connected spaces.

Let X be a pathwise connected topological space. Let G be a group acting on X by homeomorphisms and let V be a pathwise connected open subset of X whose transforms cover $X=GV$. Let E be the set of elements $\sigma\in G$ such that $V\cap\sigma V\neq\emptyset$. Let Γ be a group with one generator $[\sigma]$ for each $\sigma\in E$ and with the relations $[\sigma\tau]=[\sigma][\tau]$ whenever $V\cap\sigma V\cap\sigma\tau V\neq\emptyset$. Let $\epsilon\colon\Gamma\to G$ by $\epsilon([\sigma])=\sigma$. Macbeath's theorem asserts that ϵ is an isomorphism if $\pi_1(X)=0$. In the general case, the following result holds.