AN INVARIANT FOR ALMOST-CLOSED MANIFOLDS

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1. Let M^n be a compact, oriented, connected, n-dimensional differential manifold with ∂M (boundary M) homeomorphic to the $n-1$ sphere S^{n-1}. Then ∂M represents an element $[\partial M]$ of Γ^{n-1}, the group of differential structures (up to equivalence) on S^{n-1}. We consider the (much studied) problem of expressing $[\partial M]$ in terms of "computable" invariants of M.

Let π_{n-1} be the $n-1$ stem, $J_O: \pi_n(\mathrm{BSO}) \to \pi_{n-1}$ the classical J-homomorphism, and π'_{n-1} the cokernel of J_O. In [5], a map $P: \Gamma^{n-1} \to \pi'_{n-1}$ was defined (see below). We will define an invariant $\Delta(M)$ which is a subset of π'_{n-1} (and often consists of a single element). The main theorem states: $P[\partial M] \in \Delta(M)$.

In a strong sense, the definition of $\Delta(M)$ involves only homotopy theory. Moreover, $\Delta(M)$ seems amenable to computation by standard techniques of algebraic topology. We illustrate this below and, as applications, give explicit examples (1) of a manifold M^n, n odd, with $[\partial M] \neq 0$, and (2) of M^n, n even, with $[\partial M]$ not only $\neq 0$, but in fact with $[\partial M]$ not even contained in $\Gamma^{n-1}(\partial\pi)$, the subgroup in Γ^{n-1} of elements which bound π-manifolds. (Examples of M^n, n even, with $[\partial M] \neq 0$ are of course well known.) Other applications, and detailed proofs, will appear elsewhere.

REMARK 1. By [5], kernel $P = \Gamma^{n-1}(\partial\pi)$. If n is odd, $\Gamma^{n-1}(\partial\pi) = 0$, so P is injective, while if $n \equiv 2$ (4), kernel $P \subseteq Z_2$. If $n \equiv 0$ (4), kernel P tends to be large (but see §5).

Let BSO, BSPL, BSTop be the stable classifying spaces for orientable vector bundles, piecewise-linear ($=$PL) bundles, topological bundles. There are maps $J_G: \pi_n(\mathrm{BSG}) \to \pi_{n-1}$ ($G = O$, PL, Top) and a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \to \pi_n(\mathrm{BSO}) & \xrightarrow{f} & \pi_n(\mathrm{BSPL}) & \xrightarrow{g} & \Gamma^{n-1} \to 0 \\ \| & & \downarrow J_{\mathrm{PL}} & & \\ \pi_n(\mathrm{BSO}) & \xrightarrow{J_O} & \pi_{n-1} & \xrightarrow{q} & \pi'_{n-1} \to 0. \end{array}$$

If $z \in \Gamma^{n-1}$, define $P(z)$ as $q(J_{\mathrm{PL}}(y))$, where $g(y) = z$.

2. **On Thom complexes.** Let β be an oriented (topological) k-disk bundle over a CW-complex X, $T(\beta)$ the Thom complex. If X

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