POLYNILPOTENT GROUPS OF PRIME EXPONENT

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Let $\gamma_n(G)$ denote the nth term of the lower central series of a group G and define $\gamma_m\gamma_n(G)=\gamma_m(\gamma_n(G))$. For a fixed positive integer k define

$$f_k(1) = 1 \quad \text{and} \quad f_k(n) = f_k[n/2] + kf_k[(n+1)/2]$$

for all $n>1$. In this paper we prove

THEOREM. *Let $(m_1, \cdots, m_t)$ be a finite sequence of positive integers exceeding 1 and let G be a group of prime exponent p (p odd). Then*

$$\gamma_{r_t}(G) \subseteq \gamma_{m_1}\gamma_{m_2}\cdots\gamma_{m_t}(G),$$

where

$$r_t = m_t + \sum_{i=1}^{t-1} (m_i - 1)f_{p-2}(m_{i+1})\cdots f_{p-2}(m_t).$$

If $m_1=m_2=\cdots=m_t=2$, $r_t=1+\sum_{i=0}^{t-1}(p-1)^i$, a result of Tobin [2]. In general we have

$$\gamma_2\gamma_2\cdots\gamma_2(G) \subseteq \gamma_{m_1}\gamma_{m_2}\cdots\gamma_{m_t}(G) \quad (\gamma_2 \text{ appears } u_t \text{ times})$$

where $u_t=k+\sum_{j=1}^{t-1}(m_j-1)$ and k is the least positive integer satisfying $2^k\geqq m_t$; so that the theorem of Tobin yields

$$\gamma_{s_t}(G) \subseteq \gamma_{m_1}\gamma_{m_2}\cdots\gamma_{m_t}(G),$$

where $s_t=1+\sum_{i=0}^{u_t-1}(p-1)^i$. The bound r_t is in general far less than the known bound s_t. For instance in the very special case $(m_1, m_2, \cdots, m_t)=(2, 2^2, \cdots, 2^t)$ while $r_t<s_t$ we further observe that the degree of the polynomial r_t in p is $(t^2+t-2)/2$ as compared with 2^t-2 in s_t.

The proof of the theorem is shown to follow from the following

LEMMA.[1] *Let G be a group of prime exponent p (p odd) and let N, A, B be subgroups of G such that N is normal in G and $B\subseteq A$. Then $(N, A, B, \cdots, B)\subseteq(N, (A, B))\,(N, N)$ (B appears $p-2$ times).*

With $N=G'$ and $A=B=G$, one gets the well-known Meier-Wunderli's result that metabelian groups of prime exponent p are nilpotent of class at most p. Since

$$(\gamma_{[n/2]}(G), \gamma_{[(n+1)/2]}(G)) \subseteq \gamma_n(G) \quad \text{and} \quad \gamma_{[(n+1)/2]}(G) \subseteq \gamma_{[n/2]}(G),$$

[1] For notation and other undefined terms the reader is referred to M. Hall [1].