FREE PIECEWISE LINEAR INVOLUTIONS ON SPHERES

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If T is a piecewise linear fixed-point free involution on S^n, the orbit space $Q^n = S^n/T$ is a PL-manifold homotopy equivalent to $P_n(\boldsymbol{R}) = P^n$ [2]; the affirmative solution to the Poincaré conjecture implies that conversely for $n \neq 3, 4$ the double covering manifold of any such Q^n can be identified with S^n. Write I_n for the set of (oriented if n is even) PL-homeomorphism classes of manifolds Q^n homotopy equivalent to P^n. We will compute I_n for $n \neq 3, 4$.

Let Q^n be as above. We define a normal invariant $\eta(Q)$. Take a homotopy equivalence $h: P^n \to Q^n$ (orientation-preserving if n is odd): this is unique up to homotopy. Approximate $h \times 0$ by a PL-embedding $P^n \times Q^n \to \boldsymbol{R}^N (N > n)$; let ν^N be the normal bundle of the embedding, which exists if N is large enough [5], and $F: \nu^N \to \epsilon^N$ the fibre homotopy trivialisation induced by the homotopy equivalence [7], [10, 3.5]. Then (ν, F) induces a homotopy class $\eta(Q)$ of maps $P^n \to G/\mathrm{PL}$, which depends only on the PL-homeomorphism class of Q. We have thus defined $\eta: I_n \to [P^n, G/\mathrm{PL}]$: our description follows Sullivan [8], the main idea goes back to Novikov [6].

We next compute $[P^n, G/\mathrm{PL}]$. The homotopy groups of G/PL are known to be $\boldsymbol{Z}$ (in dimensions $4i$), $\boldsymbol{Z}_2$ (in dimensions $4i+2$), and 0 (in odd dimensions). Further, Sullivan [8] has shown that if finite groups of odd order are ignored, the only nonzero k-invariant is the first (which is δSq^2). We choose fundamental classes $x_{2i} \in H^{2i}(G/\mathrm{PL}; \boldsymbol{Z}_2)$ $(i \neq 2)$, $\alpha \in H^1(P^n; \boldsymbol{Z}_2)$. Because of the k-invariant, $[P^4, G/\mathrm{PL}] \cong \boldsymbol{Z}_4$: let y be an isomorphism. Further, denote by r the restriction $[P^{n+1}, G/\mathrm{PL}] \to [P^n, G/\mathrm{PL}]$. Then we have

LEMMA 1. *Let $i \geq 0$. Then we have bijections*

$$[P^{2i+5}, G/\mathrm{PL}] \overset{r}{\cong} [P^{2i+4}, G/\mathrm{PL}] \overset{X}{\cong} \boldsymbol{Z}_4 \oplus \sum_{1 \leq j \leq i} \boldsymbol{Z}_2,$$

where the components of X are $[y] = yr^{2i}$ and $[x_{2j+4}]$ with

$$[x_{2j+4}](f) = f^*(x_{2j+4})\alpha^{2i-2j}[P_{2i+4}].$$

Moreover, $[x_2]$ is the mod 2 *reduction of $[y]$.*

We compute the image and 'kernel' of η by surgery: in fact we have abelian groups $L_n(\boldsymbol{Z}_2^+)$ and $L_n(\boldsymbol{Z}_2^-)$ (the second referring to the non-