A GEOMETRIC INTERPRETATION OF THE KÜNNETH FORMULA FOR ALGEBRAIC \(K\)-THEORY

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1. Introduction. A Künneth Formula for Whitehead Torsion and the algebraic \(K\) functor was derived in [1], [2]. The formula reads as follows. Let \(A\) be a ring with unit and \(A[T]\) be the finite Laurent series ring over \(A\). Then, there is an isomorphism \(K_1 A[T] \cong K_1 A \oplus K_\alpha A \oplus L_1(A, T)\) where \(L_1(A, T)\) are generated by the images in \(K_1 A[T]\) of all \(I + (t^{\pm 1} - 1)\beta\), with \(\beta\) a nilpotent matrix over \(A\). On the other hand, a group \(C(A, \alpha)\) was introduced by one of the authors in his thesis [3], [4] in order to study the obstruction to fibering a manifold over \(S^1\). The group \(C(A, \alpha)\) is the Grothendieck group of finitely generated projective modules over \(A\) with \(\alpha\) semilinear nilpotent endomorphisms where \(\alpha\) is a fixed automorphism of \(A\). The structure of \(C(A, \alpha)\) suggests its close relation with the above Künneth Formula. This relation gradually became clear to us after we wrote the joint paper [5]. Since fibering a manifold over \(S^1\) is a codimension one embedding problem, one expects a good geometric interpretation of the above formula in terms of the obstruction to finding a codimension one submanifold.

In this note, we announce this interpretation which will make the relationship of [1], [2] and [3], [4], [5] even clearer. In order to put our geometric theorems in a more natural setting, we generalize the Künneth Formula to \(K_1 A_\alpha[T]\) where \(\alpha\) is an automorphism of \(A\) and \(A_\alpha[T]\) is the \(\alpha\)-twisted finite Laurent series ring over \(A\). This generalization is given in §2.

This note is an attempt to understand more about nonsimply connected manifolds and the functors \(K_\alpha, K_1\). A systematic account will appear later. We are indebted to W. Browder for calling our attention to the codimension one embedding problem.

2. The Künneth Formula for \(K_1 A_\alpha[T]\). Let \(A\) be a ring with unit. The \(\alpha\)-twisted polynomial ring \(A_\alpha[t]\) is defined as follows. Additively, \(A_\alpha[t] = A[t]\). Multiplicatively, for \(f = at^n, g = bt^m\) two monomials, \(f \cdot g = a\alpha^n(b)t^{n+m}\). Similarly, we define \(A_\alpha[T] = A_\alpha[t, t^{-1}]\). The inclusion \(i: A_\alpha[t] \subset A_\alpha[T]\) induces the exact sequence [2], [6]

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