A GEOMETRIC INTERPRETATION OF THE KÜNNETH FORMULA FOR ALGEBRAIC K-THEORY

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1. **Introduction.** A Künneth Formula for Whitehead Torsion and the algebraic K_1 functor was derived in [1], [2]. The formula reads as follows. Let A be a ring with unit and $A[T]$ be the finite Laurent series ring over A. Then, there is an isomorphism $K_1A[T] \cong K_1A \oplus K_0A \oplus L_1(A, T)$ where $L_1(A, T)$ are generated by the images in $K_1A[T]$ of all $I+(t^{\pm 1}-1)\beta$, with β a nilpotent matrix over A. On the other hand, a group $C(A, \alpha)$ was introduced by one of the authors in his thesis [3], [4] in order to study the obstruction to fibring a manifold over S^1. The group $C(A, \alpha)$ is the Grothendieck group of finitely generated projective modules over A with α semilinear nilpotent endomorphisms where α is a fixed automorphism of A. The structure of $C(A, \alpha)$ suggests its close relation with the above Künneth Formula. This relation gradually became clear to us after we wrote the joint paper [5]. Since fibring a manifold over S^1 is a codimension one embedding problem, one expects a good geometric interpretation of the above formula in terms of the obstruction to finding a codimension one submanifold.

In this note, we announce this interpretation which will make the relationship of [1], [2] and [3], [4], [5] even clearer. In order to put our geometric theorems in a more natural setting, we generalize the Künneth Formula to $K_1A_\alpha[T]$ where α is an automorphism of A and $A_\alpha[T]$ is the α-twisted finite Laurent series ring over A. This generalization is given in §2.

This note is an attempt to understand more about nonsimply connected manifolds and the functors K_0, K_1. A systematic account will appear later. We are indebted to W. Browder for calling our attention to the codimension one embedding problem.

2. **The Künneth Formula for $K_1A_\alpha[T]$.** Let A be a ring with unit. The α-twisted polynomial ring $A_\alpha[t]$ is defined as follows. Additively, $A_\alpha[t]=A[t]$. Multiplicatively, for $f=at^n$, $g=bt^m$ two monomials, $f\cdot g=a\alpha^n(b)t^{n+m}$. Similarly, we define $A_\alpha[T]=A_\alpha[t, t^{-1}]$. The inclusion i: $A_\alpha[t]\subset A_\alpha[T]$ induces the exact sequence [2], [6]

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