ON THE EXISTENCE OF EXCEPTIONAL FIELD EXTENSIONS

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Let F be a field of characteristic $p\neq 0$ and let K be an algebraic field extension of F. Let K_i denote the subfield of K of elements purely inseparable over F, K_s the subfield of separable elements, and K^n the normal closure of K/F. We say that K/F splits if $K=K_iK_s$ and following Reid's terminology in [2], K is called an *exceptional* extension of F provided $K_i=F$ and $K_s\neq K$.

LEMMA 1. *K/F splits if and only if $K_i=(K^n)_i$.*

PROOF. If K/F splits it follows easily that $K_i=(K^n)_i$. Conversely assume that $K_i=(K^n)_i$. Then K^n/K is separable normal and hence a Galois extension. Since a normal extension splits we have $K^n=(K^n)_i(K^n)_s$ and if $a\in K$, $a=\sum a_\alpha e_\alpha$ with $a_\alpha\in(K^n)_s$ and $\{e_\alpha\}$ a linearly independent set of elements of $(K^n)_i=K_i$ over F. If σ is an automorphism of K^n/K then $\sigma(a)=a$ implies that $\sum(\sigma(a_\alpha)-a_\alpha)e_\alpha=0$. But K_i and $(K^n)_s$ are linearly disjoint over F so that $\{e_\alpha\}$ is linearly independent over $(K^n)_s$. Hence $\sigma(a_\alpha)=a_\alpha$ and we have $a_\alpha\in K\cap(K^n)_s=K_s$. Thus $K=K_sK_i$.

THEOREM 2. *If K/F is a simple extension then K/F splits if and only if K^n/F is simple.*

PROOF. If K/F splits then by Lemma 1, $K_i=(K^n)_i$ and it is clear that K^n/F is also simple.

If K^n/F is simple then K/F and $(K^n)_i/F$ are simple. Let $f(X)$ be the minimum polynomial of t over F, where t is chosen such that $K=F(t)$. Then K^n is the splitting field of $f(X)$ and we have

(a) $\exp f(X)=\exp(K^n)_i$,

(b) $p^{\exp f(X)}=[K\colon K_s]$.

Since $(K^n)_i/F$ is simple it follows that $p^{\exp(K^n)_i}=[(K^n)_i\colon F]$ [3, pp. 120–123]. Hence $[K\colon K_s]=[(K^n)_iK_s\colon K_s]$ and since $K\subseteq(K^n)_iK_s$ we have $(K^n)_iK_s=K$ and $(K^n)_i=K_i$. By Lemma 1, K/F splits.

Our next lemma gives a method for constructing exceptional field extensions.

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