ON THE INTERSECTIONS OF CONES AND SUBSPACES[1]

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Introduction. Farkas' theorem [9], basic to the theory of linear inequalities (e.g. [15], [7]) and its applications (e.g. [6]), was extended to linear topological spaces in [11], [8], [3], [4], using the separation of closed convex cones and points outside them. In this note a separation argument is used to prove a theorem on the intersections of cones and subspaces in locally convex spaces, which in the finite dimensional case reduces to Farkas' theorem. This approach is similar to that in [1], [13] and [14].

Notations. Let E be a locally convex real linear topological space, E^* the space of continuous linear functionals on E. For any subset S of E let

cl(S) denote the closure of S, $S^* = \{x^*: x^* \in E^*, x^*(x) \geqq 0, x \in S\}$.

Similarly for a subset S^* of E^* let

$$^*(S^*) = \{x: x \in E,\ x^*(x) \geqq 0,\ x^* \in S^*\}.$$

If $L \subset E$ is a subspace

$$L^* = L^0 = \{x^*: x^* \in E^*,\ x^*(x) = 0,\ x \in L\}$$

and for a subspace L^* of E^*

$$^*(L^*) = {}^0(L^*) = \{x: x \in E, x^*(x) = 0,\ x^* \in L^*\}.$$

THEOREM. *Let E be a locally convex real linear topological space, L a closed linear subspace in E, C a closed convex cone in E. Then*

$$^*(L^0 \cap C^*) = \mathrm{cl}(L + C). \tag{1}$$

PROOF. Clearly $\mathrm{cl}(L+C) \subset {}^*(L^0 \cap C^*)$.

Conversely suppose there is an x_0 such that $x_0 \in {}^*(L^0 \cap C^*)$, $x_0 \notin \mathrm{cl}(L+C)$. The last fact implies that the convex compact set $\{x_0\}$ can be strictly separated from the closed convex set $\mathrm{cl}(L+C)$, e.g. [5, p. 73]. Thus there is a $y^* \in E^*$ such that $y^*(\mathrm{cl}(L+C)) \geqq 0$,

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