MODIFICATION SETS OF DENSITY ZERO[1]

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Let R, Z, T denote the real line, the integers, and the unit circle, respectively. A set $E\subset R$ will be called a *modification set in* R if to every $f\in L^1(R)$ there corresponds a singular bounded Borel measure μ on R whose Fourier transform $\hat{\mu}$ coincides with $\hat{f}$ in the complement of E. In other words, the Fourier transform of every absolutely continuous measure can be modified on E alone so that the resulting function is the Fourier transform of a singular measure. Modification sets E in Z are defined similarly: to every $f\in L^1(T)$ there should correspond a bounded singular measure μ on T whose Fourier coefficients satisfy $\hat{\mu}(n)=\hat{f}(n)$ for every integer n which is not in E.

The existence of "small" modification sets in locally compact abelian groups has been established in [1]. However, when applied to Z or R, the theorem of [1] can only yield modification sets of positive (though arbitrarily small) lower density. In the present note this result is improved to yield sets of density zero.

A set $E\subset R$ is said to have density zero if $(2t)^{-1}m(E\cap[-t, t])\to 0$ as $t\to\infty$, where m denotes Lebesgue measure. If $E\subset Z$, the requirement is that the number of elements of E in $[-N, N]$, divided by $2N$, should tend to 0 as $N\to\infty$.

THEOREM 1. *There are modification sets of density zero in* R.

THEOREM 2. *If* E *is a modification set in* R *then* $E\cap Z$ *is a modification set in* Z.

THEOREM 3. *There are modification sets of density zero in* Z.

REMARK. Modification sets can of course not be *too* small. For instance, every modification set in R has infinite measure (Plancherel); no lacunary set in Z is a modification set; no set of positive integers is a modification set (F. and M. Riesz). On the other hand, largeness is not enough: Theorem 2 shows that the complement of Z in R is not a modification set.

PROOF OF THEOREM 1. Choose integers $\lambda_1, \lambda_2, \lambda_3, \cdots$ so that $\lambda_1=10$, $\lambda_k\geqq 4\lambda_{k-1}$. Let A_k be the set of all numbers of the form

$$\pm\lambda_k+\epsilon_{k-1}\lambda_{k-1}+\cdots+\epsilon_1\lambda_1 \tag{1}$$

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