SOME TAUBERIAN THEOREMS WITH APPLICATIONS TO APPROXIMATION THEORY

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In this note we state some results, mostly without proof, concerning the comparison of integral inequalities of a certain type. As applications of these results we can deduce from a unified source theorems of approximation theory due to Jackson, Bernstein, and Zygmund, comparison theorems for moduli of smoothness of different orders, and theorems of Hardy and Littlewood and of Zygmund concerning harmonic functions. Moreover, our results yield "inverse theorems" for arbitrary integral kernels which even for many classical kernels (Fejér's, etc.) seem not to be known; thus we complete in some essential respects a program outlined by P. L. Butzer in a series of papers [1], [2], [3] (see esp. [1, p. 95]).

A preliminary version of this work giving further details may be found in the author's mimeographed lecture notes [5].[1]

1.1. We denote by R the real line, by C the class of functions real, bounded and uniformly continuous on R, and by σ a real finite signed measure on R satisfying $\sigma(R)=0$. Define for $f\in C$ and $u\geqq 0$

$$D_\sigma(f;u) = \sup_{t\in R}\left|\int f(t-uv)\,d\sigma(v)\right| \tag{1}$$

and for $t\geqq 0$,

$$\omega_\sigma(f;t) = \sup_{0\leqq u\leqq t} D_\sigma(f;u). \tag{2}$$

It is easily seen that $D_\sigma(f;u)$ (and hence also $\omega_\sigma(f;u)$) tends to zero as $u\to 0$. The primary purpose of this note is to compare (for fixed f) the rate of decrease of these functions for different choices of σ. We call D_σ, ω_σ the *σ-deviation* and *σ-modulus* of f, respectively.

1.2. EXAMPLES. (i) Take for σ the "binomial measure" β_r where r is a positive integer, i.e. β_r is the discrete measure with mass[2] $(-1)^n C_{r,n}$ at the point n ($n=0,1,\cdots,r$). The σ-modulus is then the *modulus of smoothness of order r* of the function f ([4, p. 47]). We write ω_r in place of ω_{β_r}; ω_1 is also called the *modulus of continuity* of f.

[1] For full details see the author's forthcoming paper *A Tauberian theorem related to approximation theory* in Acta Math. There a discussion of the L^p- and many-variable cases is also given.

[2] $C_{r,n}$ here denotes $r!/(n!(r-n)!)$.