INTRINSIC CHARACTERIZATION OF POLYNOMIAL TRANSFORMATIONS BETWEEN VECTOR SPACES OVER A FIELD OF CHARACTERISTIC ZERO[1]

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1. Introduction. Examples. A complex valued function u of a complex argument is a polynomial function $u(z)=\mathfrak{a}z^3+\mathfrak{b}z^2+\mathfrak{c}z+\mathfrak{d}$ of degree at most three if and only if u satisfies the inhomogeneous inclusion-exclusion identity of degree three

$$\begin{aligned} &u(\beta+\gamma+\mathfrak{d}\delta)-u(\beta-\gamma+\mathfrak{d}\delta)-u(\beta+\gamma-\mathfrak{d}\delta)+u(\beta-\gamma-\mathfrak{d}\delta) \\ &\quad=\mathfrak{d}(u(\beta+\gamma+\delta)-u(\beta-\gamma+\delta)-u(\beta+\gamma-\delta)+u(\beta-\gamma-\delta)), \end{aligned}$$

for all complex numbers β, γ, δ, $\mathfrak{d}$. The function $u(z)=z+1$ is a polynomial function of degree at most three. Suppose a real valued function t of two real arguments is Euler homogeneous of degree three. Then t is a cubic form $t(x,y)=\mathfrak{e}x^3+\mathfrak{f}x^2y+\mathfrak{g}xy^2+\mathfrak{h}y^3$ if and only if either t satisfies the heterogeneous inclusion-exclusion identity of degree three

$$\begin{aligned} &(t(\beta+\gamma+\mathfrak{d}\delta)-t(-\beta+\gamma+\mathfrak{d}\delta)-t(\beta-\gamma+\mathfrak{d}\delta)-t(\beta+\gamma-\mathfrak{d}\delta))/24 \\ &\qquad=\mathfrak{d}(t(\beta+\gamma+\delta)-t(-\beta+\gamma+\delta)-t(\beta-\gamma+\delta)-t(\beta+\gamma-\delta))/24, \end{aligned}$$

for all ordered pairs β, γ, δ of real numbers, all real numbers $\mathfrak{d}$, or t satisfies the homogeneous inclusion-exclusion identity of degree three

$$\begin{aligned} &(t(\mathfrak{b}\beta+\mathfrak{g}\gamma+\mathfrak{d}\delta)-t(-\mathfrak{b}\beta+\mathfrak{g}\gamma+\mathfrak{d}\delta)-t(\mathfrak{b}\beta-\mathfrak{g}\gamma+\mathfrak{d}\delta)-t(\mathfrak{b}\beta+\mathfrak{g}\gamma-\mathfrak{d}\delta))/24 \\ &\qquad=\mathfrak{b}\mathfrak{g}\mathfrak{d}(t(\beta+\gamma+\delta)-t(-\beta+\gamma+\delta)-t(\beta-\gamma+\delta)-t(\beta+\gamma-\delta))/24 \end{aligned}$$

for all ordered pairs β, γ, δ of real numbers, all real numbers $\mathfrak{b}$, $\mathfrak{g}$, $\mathfrak{d}$. The annihilator map $t(x,y)=0$ is a cubic form.

This paper gives the general characterization of polynomial transformations between vector spaces over a field of characteristic zero. The characterization, a generalization of A. M. Gleason's [3] and H. Röhrl's [9] recent treatment of quadratic forms, is in terms of inclusion-exclusion [4, pp. 8–10] identities. It is analogous to the characterization of a linear map v by means of the linearity identity $v(\mathfrak{a}\alpha+\mathfrak{b}\beta)=\mathfrak{a}v\alpha+\mathfrak{b}v\beta$. Constant, linear and affine maps do not fit

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