ON ASSOCIATIVE DIVISION ALGEBRAS[1]

BY A. A. ALBERT

1. Introduction. It seems highly appropriate to us that, in this address,[2] we should present to you results which represent new progress in the structure theory of associative division algebras, a field untouched for nearly thirty years, and which provided the topic of our doctoral dissertation of 1928.

As in the paper already published,[3] we shall study the structure of a central division algebra $\mathfrak{D}$, of odd prime degree p over any field $\mathfrak{F}$ of characteristic p, which has the property that there exists a quadratic extension field $\mathfrak{K}$ of $\mathfrak{F}$ such that *the algebra* $\mathfrak{D}_0=\mathfrak{D}\times\mathfrak{K}$ *is cyclic over* $\mathfrak{K}$. We shall obtain a simplified version of the JA condition that a cyclic algebra $\mathfrak{D}_0$, of degree p over $\mathfrak{K}$, shall possess the factorization property $\mathfrak{D}_0=\mathfrak{D}\times\mathfrak{K}$. We shall also derive a new *sufficient* condition that such a $\mathfrak{D}$ shall be *cyclic* over $\mathfrak{F}$, and shall present a large class of our algebras $\mathfrak{D}_0$ which satisfy this condition.

These results still leave very much open the fundamental question of the existence of noncyclic division algebras of prime degree. However, they do show that we are still far from an end to the consideration of the algebraic aspects of the problem, and are not yet really ready for the computational attack proposed in JA.

2. The field $\mathfrak{N}$. The main ingredient of our study of factorizable algebras $\mathfrak{D}_0$ over $\mathfrak{K}$ is a certain inseparable field $\mathfrak{N}$. We let $\mathfrak{K}=\mathfrak{F}(u)$ be a quadratic extension of $\mathfrak{F}$ so that we can assume that $u^2=\mu$ in $\mathfrak{F}$. Then $\mathfrak{K}$ has an automorphism $\gamma=\gamma_1+\gamma_2 u\rightarrow\bar{\gamma}=\gamma_1+\gamma_2 u$ for every γ_1 and γ_2 of $\mathfrak{F}$, and this *conjugate* operation has period two. Let g be an element of $\mathfrak{K}$ and $\mathfrak{K}(y_0)$ be a splitting field of $\mathfrak{D}_0$, where $y_0^p=g$. If there exists an element y_0^* in $\mathfrak{K}(y_0)$ such that $(y_0^*)^p=\bar{g}$, it was shown in JA that $\mathfrak{D}_0=\mathfrak{D}\times\mathfrak{K}$ implies that $\mathfrak{D}$ is cyclic over $\mathfrak{F}$. We thus assume that *the ring* $\mathfrak{K}[y_0, y_0^*]=\mathfrak{N}$, *of dimension* $2p^2$ *over* $\mathfrak{F}$ *defined by* $y_0^p=g$, $(y_0^*)^p=\bar{g}$, *is a field.*

The field $\mathfrak{N}$ now has degree p^2 over $\mathfrak{K}$, degree $2p^2$ over $\mathfrak{F}$, and $\mathfrak{K}$ is the maximal separable subfield over $\mathfrak{F}$ of $\mathfrak{N}$. The mapping over $\mathfrak{F}$ of $\mathfrak{N}$ induced by

[1] The research of this paper was supported in part by a National Science Foundation grant.

[2] This retiring Presidential Address was delivered at the Seventy Fourth Annual Meeting of the Society on January 23, 1967, in San Francisco.

[3] *New results on associative division algebras*, J. Algebra 5 (1967), 110–132. We shall refer to this paper as JA.