A FIXED POINT THEOREM OF THE ALTERNATIVE, FOR CONTRACTIONS ON A GENERALIZED COMPLETE METRIC SPACE

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1. **Summary.** The purpose of this note is to prove a "theorem of the alternative" for any "contraction mapping" T on a "generalized complete metric space" X. The conclusion of the theorem, speaking in general terms, asserts that: *either* all consecutive pairs of the sequence of successive approximations (starting from an element x_0 of X) are infinitely far apart, *or* the sequence of successive approximations, with initial element x_0, converges to a fixed point of T (what particular fixed point depends, in general, on the initial element x_0). The present theorem contains as special cases both Banach's [1] contraction mapping theorem for complete metric spaces, and Luxemburg's [2] contraction mapping theorem for generalized metric spaces.

2. **A fixed point theorem.** Following Luxemburg [2, p. 541], the concept of a "generalized complete metric space" may be introduced as in this quotation:

"Let X be an abstract (nonempty) set, the elements of which are denoted by $x, y, \cdots$ and assume that on the Cartesian product $X \times X$ a distance function $d(x, y)(0 \leqq d(x, y) \leqq \infty)$ is defined, satisfying the following conditions

(D1) $d(x, y) = 0$ if and only if $x = y$,

(D2) $d(x, y) = d(y, x)$ (symmetry),

(D3) $d(x, y) \leqq d(x, z) + d(z, y)$ (triangle inequality),

(D4) every d-Cauchy sequence in X is d-convergent, i.e. $\lim_{n,m\to\infty} d(x_n, x_m) = 0$ for a sequence $x_n \in X$ $(n = 1, 2, \cdots)$ implies the existence of an element $x \in X$ with $\lim_{n\to\infty} d(x, x_n) = 0$, ($x$ is unique by (D1) and (D3)).

This concept differs from the usual concept of a complete metric space by the fact that not every two points in X have necessarily a finite distance. One might call such a space a generalized complete metric space."

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