REPRESENTATION OF F-RINGS

BY JOHN DAUNS[1]

Communicated by R. S. Pierce, October 16, 1967

Consider a lattice ordered algebra A with identity over the rationals $\boldsymbol{Q}$; A is called an *f-ring* if $a \wedge b = 0$, $c \geqq 0$, implies that $ca \wedge b = ac \wedge b = 0$. The maximal l-ideals $\mathfrak{M}$ of A form a compact Hausdorff space in the hull-kernel topology. If A is archimedean, i.e. a so called Φ-algebra, then it is known [5] that A is isomorphic to a subalgebra of the partial algebra $D(\mathfrak{M})$ of all continuous functions f: $\mathfrak{M} \rightarrow \boldsymbol{R} \cup \{\pm \infty\}$ which are finite on a dense open set. The fact that there is a sizable theory of Φ-algebras [5], [6], [10] with no counter part for the more general class of f-rings may partly be due to the existence of this representation, $A \subseteq D(\mathfrak{M})$ for Φ-algebras, and a lack of such a representation in the nonarchimedean case. This latter representation has the defect that it is not onto. Even when $D(\mathfrak{M})$ is an algebra, A need not be all of $D(\mathfrak{M})$. Our objective is to give a representation which not only corrects this defect, but also is applicable to a wider class of f-rings. This new representation will show that the "f" in the term "f-ring" is well justified.

Define $E = \bigcup \{A/M \mid M \in \mathfrak{M}\}$; π: $E \rightarrow \mathfrak{M}$, $\pi^{-1}(M) = A/M$. Each $a \in A$ gives a map $\hat{a}$: $\mathfrak{M} \rightarrow E$, $\hat{a}(M) = a + M$. For any subset $A_1 \subseteq A$, set $\hat{A}_1 = \{\hat{a} \mid a \in A_1\}$. In order that $A \cong \hat{A}$, the condition (A) will be assumed throughout to hold

(A) $\bigcap \mathfrak{M} = \{0\}$.

Appropriate topologies can be introduced in E and $\mathfrak{M}$ making π into a structure which generalizes sheaves and fiber bundles—a so called *field*. (For a complete theory of fields, see [3].) The topologies on E and $\mathfrak{M}$ are unique in a certain well-defined sense. Let $\Gamma(\mathfrak{M}, E)$ be the l-group of all continuous cross sections σ: $\mathfrak{M} \rightarrow E$ with $\pi \circ \sigma$ the identity on $\mathfrak{M}$. Then π is continuous and $\hat{A} \subseteq \Gamma(\mathfrak{M}, E)$ is an l-subgroup. Let A^* be the subalgebra $A^* \equiv \{a \in A \mid |a| < r1, \text{some } 0 < r \in \boldsymbol{Q}\}$. Then $\Gamma(\mathfrak{M}, E)^* \equiv \{\sigma \in \Gamma(\mathfrak{M}, E) \mid |\sigma| < \hat{a} \text{ for some } a \in A^*\}$ is a convex l-subgroup of $\Gamma(\mathfrak{M}, E)$.

Although for ease of exposition, A here is the additive group of a ring, the multiplicative structure of A has not been used thus far. The above construction will be carried out more generally for an arbitrary l-group A and any set of prime subgroups $\mathfrak{M}$ with $\bigcap \mathfrak{M} = \{0\}$.

[1] Research partially supported by NSF Grant GP6219.