CHARACTERIZATIONS OF THE ESSENTIAL SPECTRUM OF F. E. BROWDER

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Let T be a densely defined closed linear operator on a Banach space X. F. E. Browder [1] has defined the essential spectrum of T, $\mathrm{ess}(T)$, to be the set of complex numbers λ such that at least one of the following conditions is satisfied:

(i) The range $\mathcal{R}(\lambda-T)$ of the operator $\lambda-T$ is not closed in X;

(ii) $\bigcup_{k\geq 0} \mathfrak{N}[(\lambda-T)^k]$ is of infinite dimension, ($\mathfrak{N}(S)$ being the null space of the operator S);

(iii) The point λ is a limit point of the spectrum of T.

In [7], M. Schechter discusses two other sets of complex numbers, $\sigma_{ew}(T)$ and $\sigma_{em}(T)$, which have also been called the essential spectrum of T (cf. [10]). He characterizes $\sigma_{em}(T)$ as the largest subset of the spectrum of T which remains invariant under compact perturbations of T. Although $\sigma_{em}(T)$ is in general a proper subset of $\mathrm{ess}(T)$, Schechter gives conditions which guarantee that $\mathrm{ess}(T)$ will remain invariant under compact (and certain other) perturbations of T. The proofs of these results usually reduce to showing that $\sigma_{em}(T)=\mathrm{ess}(T)$.

In this paper we replace Schechter's conditions on T by a condition on the perturbing operator and show that $\mathrm{ess}(T)$ is invariant under compact (and certain other) perturbations of T, provided the perturbing operators *commute* with T. We shall say that a linear operator C commutes with T if (i) the domain of C, $\mathfrak{D}(C)$, contains the domain of T, (ii) $Cx\in\mathfrak{D}(T)$ whenever $x\in\mathfrak{D}(T)$, (iii) and $TCx=CTx$ for $x\in\mathfrak{D}(T^2)$.

Following the notation and terminology of [9], we denote the dimension of the null space or *nullity* of an operator S by $n(S)$ and the codimension of the range or *defect* of S by $d(S)$. The *ascent* of S, $\alpha(S)$, is the smallest integer p such that $\mathfrak{N}(S^p)=\mathfrak{N}(S^{p+1})$, and the descent of S, $\delta(S)$, is the smallest integer q such that $\mathcal{R}(S^q)=\mathcal{R}(S^{q+1})$. (It may happen that $\alpha(S)=\infty$ or $\delta(S)=\infty$.) Suppose that λ_0 is a pole of order p of the resolvent operator $(\lambda-T)^{-1}$ and let E be the spectral projection corresponding to the spectral set $\{\lambda_0\}$. The range of E is the null space of $(\lambda_0-T)^p$ and the dimension of this space is called the *rank* of the pole λ_0.

THEOREM 1. *Let T be a densely defined closed linear operator on a*