AN EXACT SEQUENCE FOR SUBMERSIONS

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1. Introduction. Let X and B denote topological spaces. By a *topological submersion* $\pi: X \to B$, we understand a continuous map of X onto B which admits local cross-sections. With such a map one can associate an exact homology sequence

$$(1.1) \qquad \cdots \to H_q(X) \to H_q(B) \to H^r_{q-1}(X) \to H_{q-1}(X) \to \cdots$$

in which the terms $H_q(X)$, $H_q(B)$ are singular integral homology groups and the terms $H^r_q(X)$ certain "residual homology groups" to be defined. We shall examine two special cases for which the residual groups can be identified with ordinary homology groups. The first result is as follows.

THEOREM 1. *When π is the projection of an oriented k-sphere bundle, there exist canonical isomorphisms $H^r_q(X) \approx H_{q-k}(B)$ by which* (1.1) *is reduced to the Thom-Gysin sequence.*

As our second example we consider the case where π is a differentiable submersion (in the usual sense) of codimension 1. Thus we assume that X, B are C^∞-manifolds of dimensions $n+1$ and n, respectively, and that π is a regular differentiable map. Such a submersion is *orientable* if the induced line element field on X is orientable, and π will be called *simple* if $\pi^{-1}(b)$ is connected for all $b \in B$.

THEOREM 2. *When π is a simple oriented submersion of codimension 1, there exist canonical isomorphisms $H^r_q(X) \approx H_{q-1}(U)$, where U denotes the set of all $b \in B$ such that $\pi^{-1}(b)$ is compact. Thus* (1.1) *reduces to an exact sequence*

$$\cdots \to H_q(X) \to H_q(B) \to H_{q-2}(U) \to H_{q-1}(X) \to \cdots.$$

Among the consequences which ensue from this result, we mention the following

COROLLARY 1. *Let[2] $\pi: E^3 \to B$ be a simple submersion of codimension 1. Then B must be S^2 or E^2, depending on whether there does or does not exist a point $b \in B$ such that $\pi^{-1}(b)$ is compact.*

COROLLARY 2. *If $\pi: E^{n+1} \to S^n$ is a simple submersion, then $n=2$.*

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[2] E^n will denote Euclidean n-space and S^n the n-sphere.