ISOMETRIES OF L^p-SPACES ASSOCIATED WITH FINITE VON NEUMANN ALGEBRAS

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1. **Introduction.** The object of this paper is to study the isometries of the L^p-spaces, $1 \leqq p < \infty$, associated with a faithful normal semifinite trace on a von Neumann algebra M, and their connections with *-automorphisms of M (see [2], [8] for L^p-spaces, [3] for von Neumann algebras). As is well known, every *-automorphism (or *-anti-automorphism) of a finite factor M induces an L^2-isometry on M. The problem we consider is the converse: under what conditions does an L^p-isometry induce a *-automorphism? Our purpose is to provide a method for constructing *-automorphisms of von Neumann algebras.

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2. **Preliminaries.** Let M be a von Neumann algebra with a faithful normal semifinite trace ϕ. Let m_ϕ be the ideal of trace operators relative to ϕ (see [3, p. 80]). If $0<\alpha<+\infty$, m_ϕ^α denotes the ideal in M whose positive elements are the operators x^α for x a positive operator in m_ϕ. We have $m_\phi^\alpha \subset m_\phi^\beta$ if $\alpha \geqq \beta > 0$. If ϕ is finite then $M = m_\phi = m_\phi^1$ [2, p. 10]. For $1 \leqq p < \infty$ the set $m_\phi^{1/p}$ equipped with the norm $\|x\|_p = \phi(|x|^p)^{1/p}(|x| = (x^*x)^{1/2})$ is a complex normed linear space, whose completion is called the L^p-space associated with ϕ and M (see [2, pp. 23–27]). We denote this space by $L^p(\phi)$. $L^\infty(\phi)$ denotes the space M with the operator norm. It is known that $L^\infty(\phi)$ is the Banach space dual of $L^1(\phi)$ [3, p. 105], and that $L^p(\phi)$ is the Banach space dual of $L^q(\phi)$ where $1<p<\infty$ and $1/p+1/q=1$, [2, p. 27]. We use the symbol $\langle\ ,\ \rangle$ to denote these dualities and remark that if $x \in m_\phi^{1/p}$ and $y \in m_\phi^{1/q}$, then $\langle x, y\rangle = \phi(xy)$ (here, if $p=1$, $m_\phi^{1/q}$ denotes the strong closure of m_ϕ) [2, p. 27]. The space $m_\phi^{1/2}$, with the inner product $(x|y) = \phi(y^*x)$, is a pre-Hilbert space whose completion is none other than $L^2(\phi)$.

If M acts on a Hilbert space H, a closed dense linear transformation z in H is affiliated with M if $uzu^{-1} = z$ for all unitary operators u in the commutant of M (see remark following Theorem 1).

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