STRUCTURE OF CERTAIN INDUCED REPRESENTATIONS OF COMPLEX SEMISIMPLE LIE ALGEBRAS[1]

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Let $\mathfrak{L}$ be a split semisimple Lie algebra over a field Φ of characteristic zero and $\mathfrak{L}=\mathfrak{H}+\sum_{\alpha\in\Delta}\mathfrak{L}_\alpha$ be the rootspace decomposition of $\mathfrak{L}$ relative to a splitting Cartan subalgebra $\mathfrak{H}$, where the subset Δ of $\mathfrak{H}^*$ is the corresponding root-system. Fix a simple system of roots $\{\alpha_1, \alpha_2, \cdots, \alpha_l\}$, for which the positive (resp. negative) roots are denoted by Δ_+ (resp. Δ_-). For $\alpha\in\Delta$ let R_α be the Weyl reflection sending α into $-\alpha$ and fixing the elements of $\mathfrak{H}^*$ orthogonal to α with respect to the inverse Killing form $\langle\, ,\rangle$. It is given explicitly by $\lambda R_\alpha=\lambda-\lambda(h_\alpha)\alpha$ where $h_\alpha\in\mathfrak{H}$ is defined by requiring $\lambda(h_\alpha) =2\langle\alpha, \alpha\rangle^{-1}\langle\lambda, \alpha\rangle$ for all $\lambda\in\mathfrak{H}^*$. Denote the group generated by $\{R_\alpha|\alpha\in\Delta\}$ by W. We abbreviate R_{α_i} and h_{α_i} by R_i and h_i respectively. The "simple" reflections $R_1, R_2, \cdots, R_l$ are Coxeter generators of the Weyl group W. Let $\mathfrak{U}$ be the universal enveloping algebra of $\mathfrak{L}$, and $\mathfrak{U}_+$ (resp. $\mathfrak{U}_-$) the subalgebra with identity **1** generated by $\mathfrak{L}_+=\sum_{\alpha\in\Delta_+}\mathfrak{L}_\alpha$ (resp. $\mathfrak{L}_-=\sum_{\alpha\in\Delta_-}\mathfrak{L}_\alpha$).

It is an established fact that the notions of $\mathfrak{L}$-module and $\mathfrak{U}$-module are interchangeable. Here, and throughout, the word "module" is an abbreviation for the word "right-module." Our object in this paper is to study the structure of the $\mathfrak{L}$-module $\mathfrak{V}_\Lambda=\mathfrak{U}/\mathfrak{J}_\Lambda$ for arbitrary $\Lambda\in\mathfrak{H}^*$, where $\mathfrak{U}$ is regarded as a module under right-multiplication and $\mathfrak{J}_\Lambda$ is the right-ideal of $\mathfrak{U}$ (i.e., submodule of $\mathfrak{U}$) generated by

$$\mathfrak{L}_+\cup\{h-\Lambda(h)\cdot\mathbf{1}\,|\,h\in\mathfrak{H}\}.$$

It is known (cf. Cartier [**4**, p. 17–04]) that $\mathfrak{V}_\Lambda$ has a unique maximal proper submodule and hence a unique irreducible quotient-module which we denote by $\mathfrak{M}_\Lambda$. $\mathfrak{V}_\Lambda$ "admits a complete weightspace decomposition" in the sense that it is the direct sum of its weightspaces $\mathfrak{V}_{\Lambda(\lambda)}$, where for any $\mathfrak{L}$-module $\mathfrak{M}$ and any $\lambda\in\mathfrak{H}^*$ the weightspace $\mathfrak{M}_{(\lambda)}$ is defined by

$$\mathfrak{M}_{(\lambda)}=\{x\in\mathfrak{M}\,|\,xh=\lambda(h)x \text{ for all } h\in\mathfrak{H}\};$$

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